Bi-accessible and bipresentable 2-categories

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This talk is based on a joint work with Axel Osmond.

- **Bi-accessible and bipresentable 2-categories**, ArXiv:2203.07046.

Plan

1. Motivations and intuition.
2. Definition of the relevant notions.
3. Theorems and sanity check.
4. Applications.

Motivations and 2-category of interests

Develop a framework to study and classify 2-category of theories.

- **Lex**, lex categories and lex functors;
- **Reg**, regular categories and regular functors;
- **Ex**, (Barr)-exact categories and exact functors;
- **Coh**, coherent categories and coherent functors;
- $\mathbf{Pretop}_\omega$, finitary pretopoi and coherent functors.

General question: what means to *specify* a (fragment of) logic?

Spoiler

All the 2-categories above are finitely bipresentable.

Key ingredients of plain accessibility and presentability

The definitions of accessible and (locally) presentable categories are based on:

- *finite* objects;
- *directed/filtered* colimits;
- all objects are generated under finite objects via directed colimits.

Locally (finitely) presentable categories are just cocomplete finitely accessible categories.

Why do we like it as a framework?

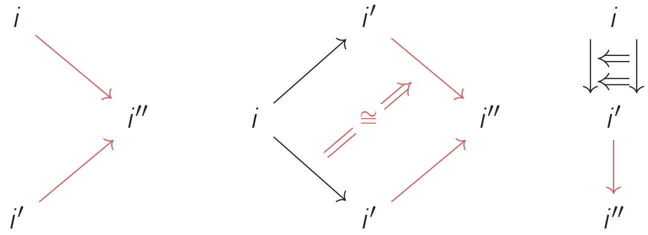
- it is tame (AFT, small object argument);
- lots of examples in nature;
- **they are sketchable**.☺.

From 1- to 2-categories

- *finite* objects;
- *directed/filtered* colimits;

Def. Kennison's Bifiltered 2-category

A 2-category I is *bifiltered* if it satisfies the following conditions:



Def. Bicompact objects

An object X in a 2-category K is bicompact if its hom-functor $K(X, -) : K \to \mathbf{Cat}$ preserve bifiltered bicolimits.

Prop. Sanity check

In Cat, finite categories are bicompact. Not all bicompact categories are finite.

Prop. Sanity check

Bicompact objects are close under weighted finite bicolimits.

Wait. Finite bicolimits?!

Def. Finite weight

A finite weight is a 2-functor $W : I \to \mathsf{Cat}$ such that I is essentially finite and $W(i)$ is a finite category for all i.

Biproducts, biequalizers and bicotensors with the arrow category suffice to construct all finite weighted bilimits.

Def. Bi-accessible 2-category

A 2-category $\mathcal{B}$ will be said *finitely bi-accessible* if

- $\mathcal{B}$ has bifiltered bicolimits,
- there is an essentially small sub-2-category $\mathcal{B}_0 \hookrightarrow \mathcal{B}$ consisting of bicompact objects such that for any B in $\mathcal{B}$ is a bifiltered bicolimit of object in $\mathcal{B}_0$.

Def. Finitely Bipresentable 2-category

Finitely Biaccessible + weighted bicolimits.

For the sake of this presentation we shall concentrate on finitely bipresentable 2-categories.

Other approaches

Other people have given alternative definitions of 2-dimensional accessibility.

- Kelly's notion of 2-accessibility is enriched, and thus too strict for our purposes. Also, in some of its definitions its treatment is very 1-dimensional.
- Bourke's 2-accessibily is based on the accessibility of the underlying category. This turns out to be surprisingly rewarding, but probably not sharp enough to precisely compute the accessibility rank of the 2-category.

Towards a reconstruction theorem

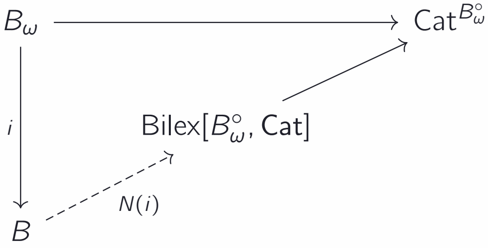
$$
\begin{array}{ccc}
B_\omega & \longrightarrow & \mathrm{Cat}^{B_\omega^\circ} \\
{\scriptstyle i}\big\downarrow & \nearrow_{N(i)} & \\
B & &
\end{array}
$$

This 2-functor preserve bifiltered bicolimits.

Can we characterize the image?

Hint

Because $N(i) = B(i-,-)$, and i preserve weighted finite bilimits, $N(i)$ lands in $\mathrm{Bilex}[B_\omega^\circ, \mathsf{Cat}]$



Does the nerve corestricts to a biequivalence of 2-categories?

Prop. Descotte-Dubuc

A prestack is bilex if and only if it is a σ-filtered bicolimit of representable objects.

- σ-filtered bicolimits are a more general notion than filtered bicolimits.
- We will not introduce σ-filtered bicolimits in this talk.
- The proposition above seems problematic. How can the nerve be surjective on objects if not all σ-filtered colimits are bifiltered?

Prop. DL-Osmond

For any σ-filtered pair (I, Σ) has a σ-cofinal 2-functor from a bifiltered 2-category.

Representation theorem

Let $\mathcal{B}$ be a finitely bipresentable 2-category. Then the binerve pseudofunctor induces a biequivalence of 2-categories with bilex pseudofunctors

$$\mathcal{B} \simeq \mathrm{biLex}[(\mathcal{B}_\omega)^\circ, \mathsf{Cat}]$$

Recognition theorem

Let $\mathcal{B}$ be locally small 2-category with weigthed bicolimits. Then the following are equivalent:

1. $\mathcal{B}$ is finitely bipresentable,
2. $\mathcal{B}$ has a strong generator $\mathcal{G} \hookrightarrow \mathcal{B}$ made of bicompact objects.

Corollary

Let T be a pseudomonad over a finitely bipresentable 2-category preserving bifiltered bicolimits. Then the category of pseudoalgebras is finitely bipresentable.

Indeed free algebras over bicompact objects for a strong generator of bicompact objects.

Adjoint functor theorem (L)

Let F be a 2-functor preserving weighted bicolimits between finitely bipresentable categories. Then it has a right biadjoint.

Adjoint functor theorems (R)

Let F be a 2-functor between finitely bipresentable categories preserving weighted bilimits and λ-bifiltered bicolimits for some λ. Then it has a left biadjoint.

Let's discuss some examples.

Example (Easy)

Cat is finitely bipresentable.

The terminal and the arrow are a strong generator.

Example (Technical)

Lex is finitely bipresentable.

The free completion under finite colimits is a finitary pseudomonad.

We now discuss **Reg, Coh, Ex, Pretopoi, (Adh, Ext)**.

On Lex Colimits

- In 2010's Garner and Lack introduced the framework of lex colimits.
- For a class of weights $\prec$ this amounts to a lax idempotent pseudomonad $\prec_l$ on **Lex**.
- Each of the 2-categories above is the categories of pseudoalgebras for a pseudomonads $\prec_l$ whose class of weights is made of finite weights.

Prop

If $\prec$ is a class of finite weights, then the 2-category of pseudoalgebras is finitely bipresentable, and moreover its bifiltered colimits are computed in Cat.

The following 2-categories are finitely bipresentable:

- **Reg**, the 2-category of small regular categories and regular functors;
- **Ex**, the 2-category of small (Barr)-exact categories and exact functors;
- **Coh**, the 2-category of small coherent categories and coherent functors;
- $\mathbf{Ext}_\omega$, the 2-category of small finitely-extensive categories and functors preserving finite coproducts;
- **Adh**, the 2-category of small adhesive categories and adhesive functors;
- $\mathbf{Pretop}_\omega$, the 2-category of small finitary pretopoi and coherent functors.