Enriched Locally Generated Categories

Ivan Di Liberti

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This talk is based on a joint work with Jiří Rosický.

- **Enriched Locally Generated Categories**, ArXiv:2009.10980.

Menu

1. Introduce the notion of locally M-generated category.
2. Discuss a sketch-like presentation: the notion of nest.
3. Generalize to enriched categories.
4. Application: Banach spaces.

Locally presentable category

A category K is locally presentable if it is a cocomplete category and has a set A of κ-presentable objects[a] such that every object is a κ-directed colimit of objects in A.

[a] for some regular cardinal κ.

... or equivalently (AR 1.11)

A category K is locally presentable if it is a cocomplete category and has a strongly generating set made of κ-presentable objects.

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Some features of Locally Presentable Categories

People like locally presentable categories.

1. many examples in nature;
2. general but *tame*;
3. small object argument;
4. crisp version of the adjoint functor theorem;
5. ...

All in all: efficient framework to do a lot (but not all) of maths.

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A focus on locally finitely presentable categories

Let's focus for a moment on locally finitely presentable category. It is

1. cocomplete;
2. has a set of *finite* objects;
3. that generates via directed colimits.

Case study: Set

- Set
- Fin, the category of finite sets.

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Case study: Grp

- Grp
- Grp_ω, the category of finitely presentable groups.

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Case study: Ban

- Banach spaces and 1-lip linear operators
- Finite dimentional Banach spaces?!

Well, not really...

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Is finite presentability the right notion of finite?

Finite dimentional Banach spaces are not finitely presentable.
Even from an enriched point of view.

Back to Gabriel and Ulmer

In their book *Lokal präsentierbare Kategorien*, Gabriel and Ulmer introduced also the notion of **locally generated category**, a variation of the notion of locally presentable category.

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Back to Gabriel and Ulmer

An object $k \in K$ is finitely generated if $K(k, -)$ preserves directed colimits **of monomorphisms**.

Locally generated category

A category K is locally generated if it is cocomplete and has a strong generator formed by λ-generated objects (and such that every such object has only a set of strong quotients).

Thm. AR

A category is locally generated if and only if it is locally presentable.

Achtung!

Locally λ-presentable $\Rightarrow$ Locally λ-generated. The viceversa is **false**.

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So...

As frameworks in which to do mathematics lg-categories are equivalent to lp-categories. But lλg-categories are more than lλp-categories.

Back to Banach spaces. AR, 2019

Let B be a finite dimensional banach space. Then the **enriched** functor $\mathrm{Ban}(B, -) : \mathrm{Ban} \to \mathrm{CMet}$ preserves directed colimits of **isometries**.

So, if we want to show that Banach spaces are somehow still *locally finitely generated* we should

- Extend the theory of locally generated category to more general classes of *monomorphisms*.
- Enrich it.

Locally generated categories

Let K be a cocomplete category with a λ-convenient factorization system (E,M). We say that K is M-locally λ-generated if it has a set A of λ-generated objects w.r.t. M such that every object is a λ-directed colimit of objects from A and morphisms from M.

Two words on history...

A similar notion was already discussed by Adamek and Rosicky.

What is this definition?

- What is a λ-convenient factorization system (E,M)?
- What is a λ-generated objects w.r.t. M?

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λ-generated objects w.r.t. M

Let $\mathcal{K}$ be a category with a factorization system $(\mathcal{E}, \mathcal{M})$. We say that an object A is λ-*generated w.r.t.* $\mathcal{M}$ if its hom-functor $\mathcal{K}(A, -) : \mathcal{K} \to \mathrm{Set}$ preserves λ-directed colimits of $\mathcal{M}$-morphisms.

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Convenient factorization system (E,M)

A factorization system $(\mathcal{E}, \mathcal{M})$ in a category $\mathcal{K}$ will be called λ-*convenient* if

1. $\mathcal{K}$ is $\mathcal{E}$-cowellpowered, i.e., if every object of $\mathcal{K}$ has only a set of $\mathcal{E}$-quotients;
2. $\mathcal{M}$ is closed under λ-directed colimits, i.e., every λ-directed colimit of $\mathcal{M}$-morphisms has the property that
    1. a colimit cocone consists of $\mathcal{M}$-morphisms, and
    2. for every cocone of $\mathcal{M}$-morphisms, the factorizing morphism is in $\mathcal{M}$.

Examples of M-generated objects

1. For the factorization system $(\mathrm{Iso}, \mathcal{K}^{\to})$, an object is λ-generated w.r.t. $\mathcal{K}^{\to}$ iff it is λ-presentable.
2. For the factorization system $(\mathcal{K}^{\to}, \mathrm{Iso})$, every object is λ-generated w.r.t. Iso.
3. For the factorization system $(\mathrm{StrongEpi}, \mathrm{Mono})$, an object is λ-generated w.r.t. $\mathrm{Mon}o$ iff it is λ-generated.

Examples of locally generated categories

1. The factorization system $(\mathrm{Iso}, \mathcal{K}^{\to})$ is convenient and $\mathcal{K}$ is locally λ-presentable iff it is $\mathcal{K}^{\to}$-locally λ-generated.
2. $\mathcal{K}$ is Iso-locally generated iff it is small.
3. In a Mono-locally λ-generated category $\mathcal{K}$ in the sense of Gabriel and Ulmer (or AR) the factorization system (strong epi, mono) is convenient.

Thm

A cocomplete category K equipped with a λ-convenient factorization system (E,M) is M-locally λ-generated iff it has a strong generator formed by λ-generated objects w.r.t. M.

Thm

Let K be a M-locally λ-generated category where (E,M) is a proper λ-convenient factorization system and let T be a monad preserving λ-directed colimits of M-morphisms. Then, assuming Vopenka's principle, the category of algebras Alg(T) is locally λ-generated.

Back to la carte

- ✓ Introduce the notion of locally M-generated category.
1. Discuss a sketch-like presentation: the notion of nest.
2. Generalize to enriched categories.
3. Application: Banach spaces.

Achtung

Don't worry, this was the longest part of the talk.

From a logical point of view: Axiomatizations.

Locally presentable categories are axiomatized by ess. algebraic theories.
Locally presentable categories are axiomatized by finite limits theories.
Locally presentable categories are sketched by finite limits sketches.
Locally presentable categories are captured by categories with finite limits.
Locally presentable categories can be reconstructed from their presentable objects.

Motto

Categories with finite limits offer a categorical presentation of essentially algebraic theories.

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Thm

Let K be a locally (finitely) presentable category. Then

$$K \simeq \mathrm{Lex}(K_\omega^\circ, \mathrm{Set}).$$

Where K_ω is the full subcategory of finitely presentable objects.

Rem

This makes sense because K_ω° is a category with finite limits.

Motto

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Goal

We wanted a similar result for locally generated categories.

Nest

A λ-nest is a category C with factorization system (E,M) such that

1. C has λ-small limits;
2. multiple pullbacks of M-morphisms.

A multiple pullback P is a limit of a diagram consisting of morphisms $f_i : A_i \to A$, $i \in I$. We can well-order I as $\{i_0, i_1, \dots, i_j \dots\}$ and form pullbacks P_j as follows: P_0 is the pullback

$$\begin{array}{ccc} P_0 & \xrightarrow{\bar{f}_0} & A_0 \\ {\scriptstyle \bar{f}_1}\downarrow & & \downarrow{\scriptstyle f_0} \\ A_1 & \xrightarrow[f_1]{} & A \end{array}$$

Then P_1 is the pullback

$$\begin{array}{ccc} P_1 & \xrightarrow{p_{01}} & P_0 \\ \downarrow & & \downarrow{\scriptstyle f_0 \bar{f}_1} \\ A_2 & \xrightarrow[f_2]{} & A \end{array}$$

We proceed by recursion and in limit steps we take limits. In this way, we transform multiple pullbacks to limits of smooth well-ordered chains (*smooth* means that in limit steps we have limits).

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Model of a nest

A model of a λ-nest is functor $C \to \mathrm{Set}$ preserving λ-small limits and multiple pullbacks of M-morphisms.

Thm

Let $(K, (E, M))$ be a M-locally finitely generated category. K°_{ω} has a structure of ω-nest and the factorization system is induced by that of K.

Thm

Let K be a M-locally finitely generated category. Then,

$$K \simeq \mathrm{Mod}(K_{\omega}).$$

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Rem

In the paper we offer a complete Gabriel-Ulmer duality for nests and locally generated categories.

Já si dám...

- ✓ Introduce the notion of locally M-generated category.
- ✓ Discuss a sketch-like presentation: the notion of nest.

1. Generalize to enriched categories.
2. Application: Banach spaces.

The notion of enriched factorization system can be defined for every enriched category.

Enriched factorization systems

For us what matters is that in a tensored V-category K, V-factorization systems are precisely factorization systems (E,M) in K_0 such that E is closed under tensors.

Convenient enriched factorization systems

A $\mathcal{V}$-factorization system $(\mathcal{E}, \mathcal{M})$ in a tensored $\mathcal{V}$-category $\mathcal{K}$ is called λ-*convenient* if it is λ-convenient as an (ordinary) factorization system.

λ-generated objects

Let $\mathcal{K}$ be a $\mathcal{V}$-category with a $\mathcal{V}$-factorization system $(\mathcal{E}, \mathcal{M})$ and λ a regular cardinal. We say that an object A is *λ-generated w.r.t.* $\mathcal{M}$ if its hom-functor $\mathcal{K}(A, -) : \mathcal{K} \to \mathcal{V}$ preserves λ-directed colimits of $\mathcal{M}$-morphisms.

Enriched M-locally λ-generated category

Let $\mathcal{K}$ be a cocomplete $\mathcal{V}$-category with a λ-convenient $\mathcal{V}$-factorization system $(\mathcal{E}, \mathcal{M})$ where λ a is regular cardinal. We say that $\mathcal{K}$ is *$\mathcal{M}$-locally λ-generated* if it has a set $\mathcal{A}$ of λ-generated objects w.r.t. $\mathcal{M}$ such that every object is a λ-directed colimit of objects from $\mathcal{A}$ and morphisms from $\mathcal{M}$.

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Thm

Assume that V is a locally λ-presentable category. Then a cocomplete $\mathcal{V}$-category $\mathcal{K}$ equipped with a λ-convenient $\mathcal{V}$-factorization system $(\mathcal{E}, \mathcal{M})$ is $\mathcal{M}$-locally λ-generated iff its underlying category $\mathcal{K}_0$ has a strong generator formed by λ-generated objects w.r.t. $\mathcal{M}$.

Fact

The notion of enriched nest, and enriched model is developed in complete analogy with the Set-based case.

When V is locally λ-presentable as a monoidal closed category one recovers the reconstruction theorem.

Thm

Assume V is locally λ-presentable as a monoidal closed category. Let K be a V-enriched M-locally λ-generated category. Then

$$K \simeq Mod_{\mathsf{V}}(K^{\circ}_{\lambda}).$$

Achtung

In the preprint we offer a full Gabriel-Ulmer duality for enriched nests and enriched locally generated categories.

Dessert

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- ✓ Generalize to enriched categories.
- 1 Application: Banach spaces.

Thm

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Back to Banach spaces

1. The CMet of complete metric spaces and non expansive maps is monoidal closed.
2. The category Ban is enriched over it.
3. (Dense maps, Isometries) is a finitely conveniente factorization system in Ban.
4. Finite dimentional Banach spaces are a strong generator formed by λ-generated objects w.r.t. isometries.

So, is Ban locally finitely generated? Well...

The is an issue

Unfortunately CMet is not finitely generated as a closed category. So we cannot apply the theorem.

But...

Ban, as an enriched category over CMet is locally finitely Isometries generated and it is axiomatized by the enriched nest of finite dimentional Banach spaces.

One must check it explicitely.