An axiomatic approach to Gabriel-Ulmer duality

Ivan Di Liberti

CT18, July 2018

Structure of the talk

- Give a definition of accessible and (locally) presentable object in a 2-category.

Structure of the talk

- Give a definition of accessible and (locally) presentable object in a 2-category.
- Cast a Gabriel-Ulmer duality for this definition.

Recap in Cat

Recap in Cat

- A is finitely accessible if $A \cong \mathrm{Ind}(G)$.

Recap in Cat

- A is finitely accessible if $A \cong \mathrm{Ind}(G)$.
- A is locally finitely presentable if $A \cong \mathrm{Ind}(G)$, for G finitely cocomplete.

Recap in Cat

- A is finitely accessible if $A \cong \mathsf{Ind}(G)$.
- A is locally finitely presentable if $A \cong \mathsf{Ind}(G)$, for G finitely cocomplete.

But most of people prefer to say:

- A is locally finitely presentable if A is a (finitely) accessibly embedded reflective subcategory of a presheaf category

$$i : A \leftrightarrows \mathsf{Set}^{G^{\mathrm{op}}} : L.$$

Recap in Cat

- A is finitely accessible if $A \cong \mathrm{Ind}(G)$.
- A is locally finitely presentable if $A \cong \mathrm{Ind}(G)$, for G finitely cocomplete.

But most of people prefer to say:

- A is locally finitely presentable if A is a (finitely) accessibly embedded reflective subcategory of a presheaf category

$$i : A \leftrightarrows \mathrm{Set}^{G^{\mathrm{op}}} : L.$$

The reason that sits at the core of the second formulation is the interplay between Ind and the presheaf construction.

Recap in Cat

- A is finitely accessible if $A \cong \mathrm{Ind}(G)$.
- A is locally finitely presentable if $A \cong \mathrm{Ind}(G)$, for G finitely cocomplete.

But most of people prefer to say:

- A is locally finitely presentable if A is a (finitely) accessibly embedded reflective subcategory of a presheaf category

$$i : A \leftrightarrows \mathrm{Set}^{G^{\mathrm{op}}} : L.$$

The reason that sits at the core of the second formulation is the interplay between Ind and the presheaf construction.

$$\mathrm{Ind}(G) \subset \mathrm{Set}^{G^{\mathrm{op}}}$$

So we isolated the essential data:

$$\mathrm{Ind}(\_) \subset \mathrm{Set}^{-^{\mathrm{op}}}$$

So we isolated the essential data:

$$\mathrm{Ind}(\_) \subset \mathrm{Set}^{-^{op}}$$

And now we need to axiomatize it.

So we isolated the essential data:

$$\mathsf{Ind}(\_) \subset \mathsf{Set}^{-^{op}}$$

And now we need to axiomatize it.

$$S \overset{Y}{\Rightarrow} P$$

- S is a KZ-monad over $\mathcal{K}$.

So we isolated the essential data:

$$\mathrm{Ind}(\_) \subset \mathrm{Set}^{\_^{\mathrm{op}}}$$

And now we need to axiomatize it.

$$S \overset{Y}{\Rightarrow} P$$

- S is a KZ-monad over $\mathcal{K}$.
- P is a KZ-monad which is also a Yoneda structure (over $\mathcal{K}$).

So we isolated the essential data:

$$\mathrm{Ind}(\_) \subset \mathrm{Set}^{\_^{\mathrm{op}}}$$

And now we need to axiomatize it.

$$S \overset{Y}{\Rightarrow} P$$

- S is a KZ-monad over $\mathcal{K}$.
- P is a KZ-monad which is also a Yoneda structure (over $\mathcal{K}$).
- Y is representably fully faithful + something else.

What’s a KZ monad?

What's a KZ monad?

Definition

A KZ monad over $\mathcal{K}$ is a lax idempotent 2-monad.

What's a KZ monad?

Definition

A KZ monad over $\mathcal{K}$ is a lax idempotent 2-monad.

A 2 monad is lax-idempotent if any algebra structure $\alpha : T(A) \to A$ is the left adjoint of the unit $A \overset{u_A}{\to} T(A)$.

What's a KZ monad?

Definition

A KZ monad over $\mathcal{K}$ is a lax idempotent 2-monad.

A 2 monad is lax-idempotent if any algebra structure $\alpha : T(A) \to A$ is the left adjoint of the unit $A \stackrel{u_A}{\to} T(A)$.

Definition

What's a KZ monad?

Definition

A KZ monad over $\mathcal{K}$ is a lax idempotent 2-monad.

A 2 monad is lax-idempotent if any algebra structure $\alpha : T(A) \to A$ is the left adjoint of the unit $A \overset{u_A}{\to} T(A)$.

Definition

A Yoneda context is a natural transformation $S \overset{Y}{\Rightarrow} P$ where

- S is a KZ monad;

What's a KZ monad?

Definition

A KZ monad over $\mathcal{K}$ is a lax idempotent 2-monad.

A 2 monad is lax-idempotent if any algebra structure $\alpha : T(A) \to A$ is the left adjoint of the unit $A \xrightarrow{u_A} T(A)$.

Definition

A Yoneda context is a natural transformation $S \overset{Y}{\Rightarrow} P$ where

- S is a KZ monad;
- P is a KZ monad which is also a Yoneda structure.

What's a KZ monad?

Definition

A KZ monad over $\mathcal{K}$ is a lax idempotent 2-monad.

A 2 monad is lax-idempotent if any algebra structure $\alpha : T(A) \to A$ is the left adjoint of the unit $A \overset{u_A}{\to} T(A)$.

Definition

A Yoneda context is a natural transformation $S \overset{Y}{\Rightarrow} P$ where

- S is a KZ monad;
- P is a KZ monad which is also a Yoneda structure.
- Y is representably fully faithful and $Y_A \cong \mathrm{Lan}_{u_A} y_A$.

What's a KZ monad?

Definition

A KZ monad over $\mathcal{K}$ is a lax idempotent 2-monad.

A 2 monad is lax-idempotent if any algebra structure $\alpha : T(A) \to A$ is the left adjoint of the unit $A \xrightarrow{u_A} T(A)$.

Definition

A Yoneda context is a natural transformation $S \overset{Y}{\Rightarrow} P$ where

- S is a KZ monad;
- P is a KZ monad which is also a Yoneda structure.
- Y is representably fully faithful and $Y_A \cong \mathrm{Lan}_{u_A} y_A$.

Definition

Given a Yoneda context $S \overset{Y}{\Rightarrow} P$ we say that A is Yaccessible if $A \cong S(G)$ for some G.

Definition

Given a Yoneda context $S \overset{Y}{\Rightarrow} P$ we say that A is Yaccessible if $A \cong S(G)$ for some G.

Definition

Given a Yoneda context $S \overset{Y}{\Rightarrow} P$ we say that A is Y-presentable if

- A is accessible;

Definition

Given a Yoneda context $S \overset{Y}{\Rightarrow} P$ we say that A is Yaccessible if $A \cong S(G)$ for some G.

Definition

Given a Yoneda context $S \overset{Y}{\Rightarrow} P$ we say that A is Y-presentable if

- A is accessible;
- $i : A \leftrightarrows P(G) : L$.

Definition

Given a Yoneda context $S \overset{Y}{\Rightarrow} P$ we say that A is Yaccessible if $A \cong S(G)$ for some G.

Definition

Given a Yoneda context $S \overset{Y}{\Rightarrow} P$ we say that A is Y-presentable if

- A is accessible;
- $i : A \leftrightarrows P(G) : L$.
- i is the Kan extenstion of its restriction to the unit of S.

Definition

Given a Yoneda context $S \overset{Y}{\Rightarrow} P$ we say that A is Yaccessible if $A \cong S(G)$ for some G.

Definition

Given a Yoneda context $S \overset{Y}{\Rightarrow} P$ we say that A is Y-presentable if

- A is accessible;
- $i : A \leftrightarrows P(G) : L$.
- i is the Kan extenstion of its restriction to the unit of S.

The last request is the translation of being *accessibly embedded*.

Definition

Given a Yoneda context $S \overset{Y}{\Rightarrow} P$ we say that A is Yaccessible if $A \cong S(G)$ for some G.

Definition

Given a Yoneda context $S \overset{Y}{\Rightarrow} P$ we say that A is Y-presentable if

- A is accessible;
- $i : A \leftrightarrows P(G) : L$.
- i is the Kan extenstion of its restriction to the unit of S.

The last request is the translation of being *accessibly embedded*.

Remark

This does not imply that $P(G)$ is accessible.

Now I want to convince you that this definition makes sense.

Now I want to convince you that this definition makes sense.

Representation Theorem

The following are equivalent:

- A is Y-presentable.
- A is Y-accessible and P-cocomplete.

Now I want to convince you that this definition makes sense.

Representation Theorem

The following are equivalent:

- A is Y-presentable.
- A is Y-accessible and P-cocomplete.

Can we cast a Gabriel Ulmer duality for such a weak notion of accessibity?

Now I want to convince you that this definition makes sense.

Representation Theorem

The following are equivalent:

- A is Y-presentable.
- A is Y-accessible and P-cocomplete.

Can we cast a Gabriel Ulmer duality for such a weak notion of accessibity?

Well, maybe one should start by recalling what Gabriel Ulmer duality is.

Gabriel Ulmer duality

There is a biequivalence of 2-categories.

$$\mathsf{Lex}^{\mathsf{op}} \leftrightarrows \mathsf{Lfp}$$

Gabriel Ulmer duality

There is a biequivalence of 2-categories.

$$\mathsf{Lex}^{\mathsf{op}} \leftrightarrows \mathsf{Lfp}$$

- Lex is the category of small finitely complete categories and functors preserving them.
- Lfp is the category of locally presentable categories and (finitely) accessible right adjoints.

Gabriel Ulmer duality

There is a biequivalence of 2-categories.

$$\mathsf{Lex}^{\mathsf{op}} \leftrightarrows \mathsf{Lfp}$$

- Lex is the category of small finitely complete categories and functors preserving them.
- Lfp is the category of locally presentable categories and (finitely) accessible right adjoints.

Can we cast a Gabriel Ulmer duality for such a weak notion of accessibity?

Gabriel Ulmer duality

There is a biequivalence of 2-categories.

$$\text{Lex}^{\text{op}} \leftrightarrows \text{Lfp}$$

- Lex is the category of small finitely complete categories and functors preserving them.
- Lfp is the category of locally presentable categories and (finitely) accessible right adjoints.

Can we cast a Gabriel Ulmer duality for such a weak notion of accessibity? Not really.

As we observed our definition of presentability does not imply that presheaf objects are presentable. In order to fix this issue we should start by understating why they are presentable in Cat.

As we observed our definition of presentability does not imply that presheaf objects are presentable. In order to fix this issue we should start by understating why they are presentable in Cat.

$$\mathsf{Set}^{G^{\mathrm{op}}} \cong \mathsf{Ind}(\widehat{G}).$$

As we observed our definition of presentability does not imply that presheaf objects are presentable. In order to fix this issue we should start by understating why they are presentable in Cat.

$$\mathsf{Set}^{G^{\mathrm{op}}} \cong \mathsf{Ind}(\widehat{G}).$$

By $\widehat{G}$ we mean the free finite colimit completion of G.

As we observed our definition of presentability does not imply that presheaf objects are presentable. In order to fix this issue we should start by understating why they are presentable in Cat.

$$\mathrm{Set}^{G^{\mathrm{op}}} \cong \mathrm{Ind}(\widehat{G}).$$

By $\widehat{G}$ we mean the free finite colimit completion of G.

Remark

In Cat there is an envelope $G \to \widehat{G}$ that fills the gap between the Ind completion and the presheaf construction.

As we observed our definition of presentability does not imply that presheaf objects are presentable. In order to fix this issue we should start by understating why they are presentable in Cat.

$$\mathsf{Set}^{G^{\mathrm{op}}} \cong \mathsf{Ind}(\widehat{G}).$$

By $\widehat{G}$ we mean the free finite colimit completion of G.

Remark

In Cat there is an envelope $G \to \widehat{G}$ that fills the gap between the Ind completion and the presheaf construction.

Definition

Given a context $S \overset{Y}{\Rightarrow} P$ a Gabriel Ulmer envelope $\widehat{(\_)}$ for Y is an addition KZ monad such that

$$S(\widehat{(\_)}) \cong P(\_)$$

We have not finished to list the axiom needed to make our version of Gabriel Ulmer duality meaningful but we announce it to itemize what we still need to comment.

We have not finished to list the axiom needed to make our version of Gabriel Ulmer duality meaningful but we announce it to itemize what we still need to comment.

Gabriel Ulmer Duality

Let $S \overset{Y}{\Rightarrow} P$ be a Yoneda context and $\widehat{(\_)}$ GU envelope for Y.

We have not finished to list the axiom needed to make our version of Gabriel Ulmer duality meaningful but we announce it to itemize what we still need to comment.

Gabriel Ulmer Duality

Let $S \overset{Y}{\Rightarrow} P$ be a Yoneda context and $\widehat{(\_)}$ GU envelope for Y. If

- $\widehat{(\_)}$ is soaking;
- S is climbable;

then

$$\mathrm{Alg}(\widehat{(\_)})^{\mathrm{op}} \cong \mathrm{Pres}(Y).$$

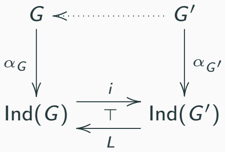
It is not worthy to tell what soaking means, but climbability is a very interesting concept for a KZ monad, which is also very natural to request.

It is not worthy to tell what soaking means, but climbability is a very interesting concept for a KZ monad, which is also very natural to request. In Cat, imagine we are in the following situation

$$\begin{array}{ccc} G & \longleftarrow\cdots\cdots & G' \\ {\scriptstyle \alpha_G}\big\downarrow & & \big\downarrow{\scriptstyle \alpha_{G'}} \\ \mathrm{Ind}(G) & \underset{L}{\overset{i}{\rightleftarrows}}\ \top & \mathrm{Ind}(G') \end{array}$$

then we know that L preserves compact objects, i.e. the dotted arrow exists.

It is not worthy to tell what soaking means, but climbability is a very interesting concept for a KZ monad, which is also very natural to request. In Cat, imagine we are in the following situation

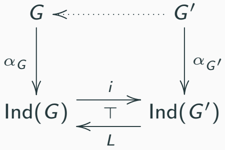


then we know that L preserves compact objects, i.e. the dotted arrow exists.

Definition

We say that S is climbable when the same property is verified.

It is not worthy to tell what soaking means, but climbability is a very interesting concept for a KZ monad, which is also very natural to request. In Cat, imagine we are in the following situation



then we know that L preserves compact objects, i.e. the dotted arrow exists.

Definition

We say that S is climbable when the same property is verified.