Givant, Morley, Zilber

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1986. **Review of Toposes, Triples and Theories by Barr and Wells**

The elements of category theory are presented with unsurpassed clarity and full motivation, and then applied to describe with equal cogency the closely related ideas of topoes, triples, and equationally defined algebraic theories. **One or two more books like this one and universal algebra might take off**.

Giancarlo Rota.

2005. **On the effective topos. Category theory mailing list.**

There is an issue involving recursivity that categorists should settle: How general is Higman's theorem? In group theory the word problem [...] is equivalent to the purely algebraic one of whether the given group can be embedded as a subgroup of a finitely presentable one. For which other algebraic categories is the same statement true? or is it possibly true for the category of single-sorted algebraic theories?

Bill Lawvere.

Agenda

- Present a cluster of problems motivated by universal algebra;
- Convince you that there is something to learn;
- Provide enough historical framing to appreciate them.

Rings and their modules

Let R be a unital commutative ring. TFAE:

1. R is a field;
2. every R-module is free.

Key ingredients in (1) $\Rightarrow$ (2)

- Let M be an R-module. For $A \subset M$ we have $\mathrm{Span}(A)$.
- Transfinite induction. $A_0 = \emptyset$. $A_n = A_{n-1} \cup \{x\}$ with $x \in M - \mathrm{Span}(A_n)$.
- Terminate at some ordinal λ. A_λ is a **basis** for M.

Thm.

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Reformulation

Consider the monad $R[-] : \mathsf{Set} \to \mathsf{Set}$. TFAE.

- R is a field;
- The comparison functor $U : \mathsf{Kl}(R[-]) \to \mathsf{Alg}(R[-])$ is an equivalence of categories.

For the sake of this talk, a monad such that U is an equivalence of categories will be called **stable**.

Question

How do *finitary* stable monads on Set look like ? Should we care?

Examples of stable monads

- Idempotent monads are stable. But there are very few.
- The maybe monad ($X \mapsto X_*$) is stable! $\mathsf{Alg}(T) = \mathsf{Set}_*$.
- any other idea ...?

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- well...

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- there aren't.

Question

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Examples of stable monads

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- The maybe monad $(X \mapsto X_*)$ is stable! $\mathsf{Alg}(T) = \mathsf{Set}_*$.
- kind of.

Thm. (Givant 1970)

There are precisely 4 finitary stable monads:

- The identity;
- The maybe monad;
- $R[-]$ for R a division ring (a noncommutative field);
- $Aff_R[-]$, the free affine space over a division ring.

Givant is a universal algebraist, his result is formulated in terms of varieties of algebras.

The theorem itself is a quite strange serendipity. No?

- Set_* is already considered since quite some time $\mathsf{Mod}(\mathbb{F}_1)$, the category of modules over the *field with one element*.
- Thus, for some reason, when T is stable, $T(1)$ is (a) forced to be a kind of ring (b) contains enough information to recover the whole monad.

We need a deeper exploration: pointed sets

How can we show, in the most conceptual way possible, that the maybe monad is stable?

Spans and closure operator

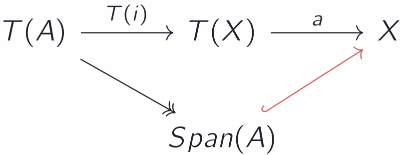
Consider a pointed set (X, x_0). For a subset $A \subset X$ define $\mathsf{Span}(A)$ the smallest subalgebra of (X, x_0) containing A . Of course, given the simplicity of this monad, this is just $(A \cup \{x_0\}, x_0)$.

Extraction of a basis

Now we show that every algebra is free. We do as in the case of vector spaces. In this case, the basis is precisely given by $X - \{x_0\}$, as it should be.

Can we always define a notion of Span?

Yes. Let T be a monad on **Set** and (X, a) be an algebra. Then, for $i : A \subset X$ consider



Similar ideas (but never quite the same) have been considered by *Tholen et al* in the topic of **closure operators**. This notion coincides with the previous one in the case of vector spaces and pointed sets.

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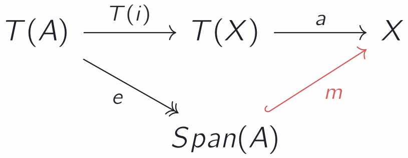

Generators and linear independence

- $A \subset X$ is a generator if $\mathsf{Span}(A) \to X$ is in $\mathcal{E}$.
- The elements of A are linearly independent if e is in $\mathcal{M}$.

Can we always define a notion of Span?

Yes. Let T be a monad on Set and (X, a) be an algebra. Then, for $i : A \subset X$ consider

$$\begin{array}{ccccc} T(A) & \xrightarrow{T(i)} & T(X) & \xrightarrow{a} & X \\ & {\scriptstyle e}\searrow\!\!\!\!\searrow & & \nearrow{\scriptstyle m} & \\ & & Span(A) & & \end{array}$$

Generators and linear independence

Other notions of basis have been proposed, for example by *Jacobs*. It's hard to tell whether this technology is sharp enough.

So it seems we can generalize the main ingredients of the proofs.

Prop

Let T be a finitary monad on Set, then

- $\mathrm{Span}_X : \mathcal{P}(X) \to \mathcal{P}(X)$ is a closure operator (monad) and preserve directed colimits.

Moore Toys

A Moore toy $(X, \mathbf{cl})$ is a set equipped with a closure operator $\mathbf{cl} : \mathcal{P}(X) \to \mathcal{P}(X)$. A morphism of Moore toys is a set function commuting with the closure operators.

- Top $\to$ Moo mapping $(X, \tau) \mapsto (X, \mathbf{cl}_\tau)$.The functor is ff by Kuratowski formulation of continuity.(Is it actually coKZ reflective?)
- Grp $\to$ Moo mapping $G \mapsto (G, \langle - \rangle)$.Faithful.
- RMod $\to$ Moo mapping $M \mapsto (M, \mathrm{Span})$.Faithful.
- Mod($\mathbb{T}$) $\to$ Moo mapping $M, \mapsto (M, \mathbf{dcl})$.Faithful.

Prop

Let T be a finitary monad on Set, then we have a faithful functor,

$$\mathsf{Alg}(T) \to \mathsf{AMoo}$$

with the category of *algebraic* Moore toys, i.e. **cl** preserves filtered colimits.

In model theory a special class of Moore toys (pregeometries) has been studies by Zilber. A pregeometry is a Moore toy verifying the Steinitz exchange property. This topic goes under the name of *geometric stability theory*.

Zilber's Motto

When a pregeometry (M, cl) is nice enough , we can find a field $\mathbb{F}$, a structure of abelian group on M and an action of the field such that

$$(M, \mathsf{cl}) \cong (M, \mathsf{Span})$$

That is, M must be a kind of vector space.

Strategy

property of $\mathsf{Alg}(T) \rightsquigarrow$ property of $\mathsf{AMoo} \rightsquigarrow$ construction of *rigid*/*simple* algebraic data $\rightsquigarrow$ back to the monad

- Givant's idea are somewhat related to this strategy.
- The papers are VERY FAR from this narrative. A categorical account is far from being there.

... Morley?

The question

what kind of finitary monads are such that every algebra is free?

fits in the very general landscape of

what kind of first order theories are such that there is only one model per each cardinality?

Geometric stability theory emerged as a branch of this circle of thoughts, contaminated by ideas coming from minimality, algebraic geometry and ring theory.

Morley

In 1960's Morley proved that if a first order has only one model (is categorical) in some cardinal λ, then it has only one model in all cardinals above $\aleph_1$.

This shows a very strong rigidity of theory with respect to their models, at least in this behavior, as soon as there is only one model in some cardinality, all the cardinals obey to the same order.

Reverse Wrap up

- We are far from having a complete categorical understanding of Morley's categoricity result, and its generalization the Shelah's conjecture for Abstract Elementary classes,
- Beke, Vasey, Rosicky, Lieberman have done several steps in this direction, now we have a very semantic and somewhat partial understanding of the situation,
- Zilber offers a linear-friendly and geometric interpretation of categoricity for first order logic via geometric stability,
- Givant only studies categoricity in in the case of universal algebra. This is of clourse the most approachable framework, and having a neat understanding of this case could inspire new ideas for all its generalizations.
- The whole talk was reading this item list bottom up.

Completely insufficient bibliography, email me for the rest

- **Zilber**, Hereditarily Transitive Groups and Quasi-Urbanik Structures;
- **Givant**, Universal classes categorical or free in power;
- **Morley**, Categoricity in power;
- **Rosicky**, Concrete categories and infinitary languages;
- **Hyttinen** and **Kangas**, Categoricity and universal classes;
- **Campion**, Varieties where every algebra is free, MathOverflow.
- **Rosicky**, Categories, saturation and categoricity;
- **Tholen** and **Dikranjan**, Categorical Structure of Closure Operators;
- **Wolff**, Monads and Monoids on Symmetric Monoidal Closed Categories;
- **Jacobs**, Bases as coalgebras;