The shape of water

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Thm. (Whitehead '49)

The functor

$$\mathrm{ho}(\mathrm{CW}_*) \xrightarrow{(\pi_n)_{n\in\mathbb{N}}} \mathrm{Grp}^{\mathbb{N}}$$

reflects isomorphisms (is conservative).

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Observe that we did not give a precise notion of *algebraic category*. In fact, we aim to replace A with the category of sets. We feel free to make this choice because, whatever algebraic means, the category of sets will be an algebraic category, and any algebraic category has a faithful and conservative functor to Set.

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Thm. (Freyd '70)

There is no faithful functor $\mathrm{ho}(\mathrm{CW}_*) \to \mathrm{Set}$.

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Let $\mathcal{C}$ be a category with finite limits. $\mathcal{C}$ is concrete (i.e. has a faithful functor to Set) if and only if it is regular well-powered.

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Thm. Loregian, DL '17

Let $\mathcal{M}$ be a pointed model category; if there exist an index $n \in \mathbb{N} \geq 1$ and a 'weak classifying object' for the functor $\pi_n : \mathcal{M} \to \mathsf{Grp}$, then $\mathsf{ho}(\mathcal{M})$ is not concrete.

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By *weak classifying object* we mean a very weak notion of Eilenberg-Mac Lane spaces that we introduced in the paper. There is no need to specify that Eilemberg-Mac Lane like constructions occur all the time in model categories.

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The following are all examples of weak classifying objects for a model category:

1. A section for π_n;
2. A faithful left adjoint for π_n;
3. A full right adjoint for π_n.

In the direction of conservativity

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1. In his paper Freyd explicitly provides the conservative functor $\mathcal{C} \to \text{Set}$. Even in the most elementary case, how does this functor look like? Might it be interesting to actually use it independently from its abstract application?
2. It is possible to rephrase Freyd's theorem saying that if $\mathcal{C}$ is enriched over Set, then there is a conservative functor to Set. Is it possible to state the same theorem for categories enriched over an elementary topos?

In the direction of faithfulness

Given two categories A and B we say that A is more complicated than B ($A \geq B$) if there is a faithful functor from B to A.

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1. Is this order even *total*?
2. Is $\text{ho}(\text{CW}_*)$ the most complicated category?
3. Does any *homotopical* category have a faithful functor into $\text{ho}(\text{CW}_*)$?