Call doctrines by your name

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This talk is based on a preprint and an ongoing project.

- **Logic and Concepts in the 2-category of Topoi**,
  ArXiv:2504.16690. j/w *Lingyuan Ye*.
- **From lax idempotent pseudomonads to Lawverian doctrines**,
  work in progress, j/w *J. Emmenegger* and *J. Wrigley*.

Plan

1. Motivations:
   - what's a *doctrine* in categorical logic?
   - what's a fragment of geometric logic?
2. Kan injectivity and semantic prescriptions
3. Syntactic categories and syntactic sites
4. Kock-Zoberlein doctrines on Lex
5. Classifying topoi and Diaconescu
6. Completeness theorems and open problems
7. From Kock-Zoberlein doctrines on Lex to Lawvererian doctrines

An elephant in the room of categorical logic

What's a *doctrine* in category theory?

- It's a *Kock-Zoberlein* doctrine, i.e. a lax-idempotent pseudomonad.
- It's a Lawvere-style (hyper)doctrine $\mathcal{P} : C^{\mathrm{op}} \to \mathsf{Pos}$.
- it's a fragment of predicate logic.
- It's a type of topos associated to a syntactic category/site.

Examples of doctrines

(essentially) algebraic, regular, coherent,disjunctive, geometric...

Question: Can we find unity in this picture?

Is it a coincidence that all these objects share the same name?Can we (a) provide explicit constructions to translate between these theories and (b) give a satisfying and precise notion of doctrine that unifies these representations? **Yes.**

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More foundationally

Can we give a (mathematical) definition of *fragment of geometric logic* that has as *features* all these described elements?**Yes.**

But practically, why should we care?

Structural/modular results about logics:which logics admit a Craig interpolation theorem?Can we provide a categorical version of Lindstrom theorem?

Different fragments have different semantics properties

- Essentially algebraic $\rightsquigarrow$ any (co)limit of models.
- Regular $\rightsquigarrow$ products and directed colimits of models.
- Disjunctive $\rightsquigarrow$ connected limits and directed colimits of models.
- First order/coherent $\rightsquigarrow$ ultraproducts and directed colimits of models.
- Geometric $\rightsquigarrow$ directed colimits of models.

Idea! Semantic prescriptions

A (fragment of geometric) logic is a collection of prescribed properties that categories of models of theories in such fragment will enjoy.

Let's make an example

Let $\mathcal{E}$ be a topos. The following are equivalent (up to retract):

- $\mathcal{E}$ classifies an essentially algebraic theory.
- For every geometric morphism $f : \mathcal{X} \to \mathcal{Y}$, the right Kan extension above exists.

$$\begin{tikzcd} \mathcal{X} \arrow[r, "x"] \arrow[d, "f"'] & \mathcal{E} \\ \mathcal{Y} \arrow[ur, dashed, "\mathrm{ran}_f x"'] & \end{tikzcd}$$

(Weak Kan Injectivity)

In the recent paper **KZ monads and Kan Injectivity** by Sousa, Lobbia and DL this behaviour is called Weak Kan Injectivity (with respect to a morphism f).

Example

For $\mathcal{E}$ a topos, if we want to prescribe its category of models (in Set) to have all limits over diagrams of shape I, it's enough to require Kan injectivity with respect to,

$$\begin{tikzcd} \mathsf{Set}^I \arrow[r, "x"] \arrow[d, "\Gamma"'] & \mathcal{E} \\ \mathsf{Set} \arrow[ur, dashed, "\mathsf{ran}_f x"'] & \end{tikzcd}$$

Thm. DL, 2022

If a topos is right Kan injective with respect to the morphisms below, its category of points is equipped with an ultrastructure.

$$\begin{tikzcd} \mathsf{Set}^X \arrow[r, "x"] \arrow[d, "\iota_X"'] & \mathcal{E} \\ \mathsf{Sh}(\beta(X)) \arrow[ur, dashed, "\mathsf{ran}_f x"'] & \end{tikzcd}$$

Definition: Fragment of geometric logic

A logic is a class of geometric morphisms $\mathcal{H}$. A topos *formally belongs* to a logic if it is weakly right Kan injective with respect to all geometric morphisms in $\mathcal{H}$. These are collected in the 2-category $\mathsf{WRInj}(\mathcal{H})$.

Example

- when $\mathcal{H}$ is the class of all geometric morphism, one shows that $\mathcal{E} \in \mathsf{WRInj}(\mathcal{H})$ iff is a retract of a presheaf topos over a lex category.
- when $\mathcal{H}$ is empty, every topos is in $\mathsf{WRInj}(\mathcal{H})$
- when $\mathcal{H}$ is given by $\mathsf{Set}^X \to \mathsf{Sh}(\beta(X))$, $\mathsf{WRInj}(\mathcal{H})$ contains all coherent topoi.

Remark

Since Set^C for C a lex category is weakly right Kan injective with respect to all geometric morphisms, it is in particular in all logics.Hence, $\mathsf{Set}[\mathbb{O}]$ is in $\mathsf{WRInj}(\mathcal{H})$ for all $\mathcal{H}$.

Syntactic categories and syntactic sites

We have a 2-functor

$$\mathsf{Syn} : \mathsf{WRInj}(\mathcal{H})^{\mathsf{op}} \to \mathsf{Lex}$$

$$\mathcal{E} \mapsto \mathsf{WRInj}(\mathcal{E}, \mathsf{Set}[\mathbb{O}]).$$

$\mathcal{H}_\emptyset$ $\mathsf{Syn}^{\mathcal{H}_{\mathsf{eth}}}$ is the forgetful functor

$$\mathsf{U} : \mathsf{Topoi}^{\mathsf{op}} \to \mathsf{LEX},$$

$\mathcal{H}_{\mathsf{all}}$ For a free topos $\mathsf{Psh}(\mathcal{C})$, $\mathsf{Syn}^{\mathcal{H}}(\mathsf{Psh}(\mathcal{C}))$ coincides precisely with the full subcategory of representables, a.k.a. $\mathcal{C}$ itself.

$\mathcal{H}_\beta$ For a free topos $\mathsf{Psh}(\mathcal{C})$ $\mathsf{Syn}^{\mathcal{H}_\beta}(\mathsf{Psh}(\mathcal{C}))$ coincides precisely with the full subcategory spanned by the coherent completion of $\mathcal{C}$.

Construction: The Beth (relative) pseudomonad associated to a logic

For $\mathcal{H}$ a logic, consider the composition below.

$$\begin{tikzcd} \mathsf{lex} \arrow[r, "\mathsf{Psh}"'] \arrow[rr, bend left, dashed, "\mathsf{T}^{\mathcal{H}}"] & \mathsf{WRInj}(\mathcal{H})^{\mathrm{op}} \arrow[r, "\mathsf{Syn}^{\mathcal{H}}"'] & \mathsf{LEX} \end{tikzcd}$$

Examples

- $\mathcal{H}_{\emptyset}$ $\mathsf{Alg}(T^{\mathcal{H}})$ is the 2-category of infinitary pretopoi
- $\mathcal{H}_{\mathrm{all}}$ $\mathsf{Alg}(T^{\mathcal{H}})$ is lex itself.
- $\mathcal{H}_{\beta}$ $\mathsf{Alg}(T^{\mathcal{H}})$ is the 2-category of Pretopoi

Achtung!

The last result hinges on Makkai's conceptual completeness and we do not have a non-semantic proof of this result.

Construction: the classifying topos of an algebra

Every algebra can be equipped with a canonical structure of site, on which we can take sheaves.

$$\begin{tikzcd} \mathsf{Alg}(\mathsf{T}^{\mathcal{H}})_{\mathsf{M}} \arrow[rr, "\mathsf{Cl}"] \arrow[dr, "J^{\mathcal{H}}"'] & & \mathsf{Topoi}^{\mathsf{op}} \\ & \mathsf{SITES}_{\mathsf{M}} \arrow[ur, "\mathsf{Sh}"'] & \end{tikzcd}$$

Theoremm: Diaconescu

We have a relative pseudoadjunction as below,

$$\begin{tikzcd} \mathrm{alg}(T^{\mathcal{H}}) \arrow[r, "\mathrm{Cl}"] \arrow[d] & \mathrm{Topoi}^{op} \arrow[dl] \arrow[l, phantom, shift right=3, "\perp"] \\ \mathrm{Alg}(T^{\mathcal{H}}) & \end{tikzcd}$$

$$\begin{tikzcd} \mathrm{lex} \arrow[r, "\mathrm{Cl}"] \arrow[d] & \mathrm{Topoi}^{op} \arrow[dl] \arrow[l, phantom, shift right=3, "\perp"] \\ \mathrm{LEX} & \end{tikzcd} \qquad \begin{tikzcd} \mathrm{Topoi}^{op} \arrow[r, "\mathrm{Cl}"] \arrow[d] & \mathrm{Topoi}^{op} \arrow[dl] \arrow[l, phantom, shift right=3, "\perp"] \\ \mathrm{Pretopoi}_{\infty} & \end{tikzcd}$$

Example

When $\mathcal{H}$ is the class of β-maps, we obtain the classifying topos over a pretopos, which by Makkai's theorem is 2-fully faithful.

Definition

A logic $\mathcal{H}$ enjoys conceptual completeness if the 2-functor exhibiting conceptual soundness $\mathsf{Alg}(\mathsf{T}^{\mathcal{H}})^{\mathsf{op}} \to \mathsf{WRInj}(\mathcal{H})$ is in fact 2-fully faithful.

Question

What logics $\mathcal{H}$ enjoy conceptual completeness?

Toy theorem (DL-Ye): propositional boost

If a fragment of geometric logic admits a completeness theorem over Set-models for its propositional truncation, then it admits a completeness theorem also for its predicate version.

Achtung!

Of course this theorem ought to be true, but until recently we did not even have the language to state (especially in categorical language).

Recap

For a class of geometric morphisms (semantic prescription) we found a way to build a syntactic category (and a syntactic site), which yields KZ doctrine over lex.
The algebras for such doctrine all admit a classifying topos, recovering many usual construction in categorical logic, including variations of Diaconescu's theorem.

Question

What about Lawvererian doctrines?

Construction, DL-Emmenegger-Wrigley

For T a KZ doctrine over lex, one can build a KZ doctrine $\mathsf{T}^{\mathsf{fbr}}$ over PDoc in such a way that:

- when T is the presheaf construction $\mathsf{T}^{\mathsf{fbr}}$ is the free locale completion.
- when T is the free coherent category, $\mathsf{T}^{\mathsf{fbr}}$ is the coherent completion of a primary doctrine.