# RINGS AND MODULES

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This note is going to summarize the content of the first lesson of tutoring on the course Rings and modules. Also, attached in the end, there is an exercise sheet.

Contents



## 1. Class

I would like to start by recalling some definitions.

**Definition 1.** *Let $A$ is a ring and $1$ its multiplicative identity. A left $A$-module $M$ consists of an abelian group $(M, +)$ and an operation $\cdot : A \times M \to M$ such that for all $a, b$ in $A$ and $m, n$ in $M$, we have:*

- $a \cdot (m + n) = a \cdot m + a \cdot n$
- $(a + b) \cdot m = a \cdot m + a \cdot n$
- $(ab) \cdot m = a \cdot (b \cdot m)$
- $1 \cdot m = m$.

**Remark 2.** Let's be honest, rings will be most of the time unital and commutative. When rings are unital, rings morphisms will preserve units. In the case of commutative rings there is no difference between right and left modules, thus we will not care.

**Definition 3.** *A submodule $N \subset M$ of the $A$-module $M$ is a subset, which is an $A$ module such that the inclusion is a module homomorphism.*

**Remark 4.** A course on linear algebra suggests that these are just the definitions of vector space and subspace, where we allow $A$ to a be a ring, with no restriction on it.

Thus it pops out as a very natural question, which I would like to answer today, at least partially: *How far is module theory from linear algebra?*

We shall start by giving some very natural examples of modules, to get in touch with some of the phenomenology that this definition is allowing.

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### 1.1. Examples of modules.

**Example 5** (A)**.** If $A$ is a ring, clearly $A$ has an $A$-module structure, by putting:

$$a \cdot m := am.$$

Where on the right side we mean the multiplication of the ring. What are the submodules of this module? It is quite easy, almost tautological in fact to, to observe that submodules of $A$ with this $A$-module structure are precisely ideals of $A$.

In the case of linear algebra, this is precisely the vector space of dimension 1 over the field. We shall observe that a field has no non trivial ideals, thus there are no non trivial submodules. This is the first, important difference between linear algebra and general module theory.

**Example 6** (A/I)**.** Let $A$ be a ring and $I$ be an ideal of $A$, then ${}^{A}\!/_{I}$ has a natural module structure:

$$a \cdot [m] := [am],$$

which makes sense because $a \cdot I \subset I$.

This phenomenon has not role in the play in linear algebra, because we will not get anything interesting, we will not find non trivial ideals of $A$.

Some concrete instances of this example are the following:

- $\mathbb{Z}_2$ is a $\mathbb{Z}$-module.
- $\mathbb{K}[x]/(f)$ is a $\mathbb{K}[x]$-module.

**Example 7** (Subrings)**.** Let $A \subset B$ two rings such that the inclusion is a ring homomorphism, then $B$ has a structure of $A$-module by putting:

$$a \cdot b := ab.$$

Also, a ring morphism $A \xrightarrow{f} B$ induces an $A$-modules structure on $B$ by putting

$$a \cdot b := f(a)b.$$

**Question 8.** Can you see how to make this last construction as a special case of the ones before?

Examples of this construction are the following:

- $A[x]$ is a $A$-module.
- $\mathbb{K}[x]$ is a $\mathbb{K}$-module (vector space).

**Example 9** ($\psi$)**.** Now we come to a juicy example. Let $V$ be a vector space over the field $\mathbb{K}$ and $\psi$ be an endomorphism of $V$, then $V$ has a natural structure of $\mathbb{K}[x]$-module as follows:

$$p(x) \cdot v := p(\psi)(v).$$

Here $\psi^n$ is the composition $n$ times of $\psi$. This is the first, not completely trivial example we gave.

Can we imagine a submodule of $V$? Let's make some observations.

- It is going to be a subspace $W \subset V$
- $x \cdot W \subset W$ implies $\psi(W) \subset W$.

It is not hard to notice that these conditions are also sufficient to be a submodule, so that submodules of $V$ with this modules structure are precisely invariant subspaces under the action of $\psi$.

I hope that one can see the expressiveness of module theory, one can encode the notion of being an invariant subspaces in the one of being a submodules of a certain module structure.

At some point the following sentence will be clear to you: the Jordan normal form is a decomposition in indecomposables of $V$ under this module structure.

### 1.2. **Linear algebra and module theory.**

1.2.1. *A bad news.* We come back to our main question. How far is module theory from linear algebra? The bad new is that it is quite far, the good news is that nowadays we have a fairly precise idea of how far they are. Here I mean that we know which theorem and which intuitions we can transport from linear algebra to module theory.

Today we will look at the biggest rift, the one any story should start from.

**Theorem 1.1** (Existence of a basis)**.** *Let $V$ be a vector space over the field $\mathbb{K}$, then there exist an isomorphism*

$$V \to \mathbb{K}^{\Gamma},$$

*for a suitable $\Gamma$.*

This theorem also established a correspondence between vector morphisms and matrices and yelds the most effective tool in linear algebra. Moreover the theorem asserts that there are not so many behaviours for a vector space, there is a special structure, the one on the right side, that everyone is just copying.

**Remark 10.** Hey, by $\mathbb{K}^{\Gamma}$ we mean that the sequences are definitively constantly 0! This one I chose for this object is a terrible notation, but it is the best we can afford today. I am sorry for that.

**Question 11.** Why did I name this theorem *existence of a basis*?

We shall show that this theorem is false in the case of general modules.

**Proposition 1.2.** *Let $A$ be a ring with a proper ideal, then ${}^{A}\!/_{I}$ is never isomorphic to $A^{\Gamma}$ for any $\Gamma$.*

*Proof.* We argue by contradiction. Call $\psi : A^{\Gamma} \to {}^{A}\!/_{I}$ the iso. The set $I \cdot A^{\Gamma}$ is a proper submodule of $A^{\Gamma}$. Let's give a look to what it corresponds to.

$$\psi(I \cdot A^{\Gamma}) = I \cdot \psi(A^{\Gamma}) = I \cdot {}^{A}\!/_{I} = 0$$

and this is just absurd, $\psi$ cannot map a non trivial submodule to a trivial one. □

**Example 12.** $\mathbb{Z}_2$ cannot be isomorphic to $\mathbb{Z}^n$ for any $n$.

Let us collect ideas. In linear algebra we have a standard shape for vector spaces, which often makes things easier. There is no standard shape in module theory. But...

1.2.2. *The apple never falls far from the tree.* Prof. Rosický will call modules of the shape $A^{\Gamma}$ *free modules.* They are very important and you will see them often in this course. In the last section we stated that every vector space is free while some modules are not. What can we save in this dark night in which we lost our matrix representation?

**Proposition 1.3.** *Any module is a quotient of a free one.*

*Proof.* Let $M$ be the module. Consider the free module $A^M$. There is a natural surjective morphism

$$A^M \twoheadrightarrow M.$$

□

**Example 13.** $\mathbb{Z}_2$ is a quotient of $\mathbb{Z}$.

Many ideas of module theory, such as Smith normal form or free/projective resolution have their roots in the existence of this cover, which establishes that, if everything is not free, then, ok, everything can be covered by something which is free.

**Remark 14.** Can we say that any theorem which is true for vector spaces is true for free modules? The answer to this question, which is unclear in its statement is *no* but the intuition that we have for vector spaces is good when thinking to free modules and last theorem proves that we can push this intuition along the quotient and *hope for the best!* This, in fact, happens quite often.

## 2. Exercises

Pay attention, exercises labelled by the tea cup ☕ may not be incredibly challenging, even not challenging, but it is important keep them in mind, so take your time when solving them and be careful to find a formal and correct solution. Exercises labelled by the danger international sign ⚠ are very challenging.

**Exercise 1.** ☕ Take a book of linear algebra and read the proof of existence of a basis. Where is it using the hypothesis of having a field? Can you see how this is linked to Proposition 1.2?

**Exercise 2.** Can you write down the precise module structure I am using on $A^\Gamma$? What is this *natural morphism* in 1.3?

**Exercise 3.** Show that a $\mathbb{Z}$-module is just a commutative group.

**Exercise 4.** Show that if $G$ is a commutative group then the set of group endomorphisms $\text{End}(G)$ of $G$ has the structure of a ring with addition

$$(f+g)(a) = f(a) + g(a)$$

and with multiplication given by composition.

**Exercise 5.** ☕ Let $A$ be a ring and $G$ be an abelian group. Show that ring homomorphisms

$$A \to \text{End}(G)$$

are in bijection with $A$-module structures on $G$.

**Exercise 6.** In the remark 2 we said that under the assumption of commutativity one can identify left and right modules. What do we mean?! What happens when the ring is not commutative?!

**Exercise 7.** ⚠ $\mathbb{Q}$ is a $\mathbb{Z}$-module with the very natural structure of Example 7. Can you tell if it is it free?!