RINGS AND MODULES

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This note is going to summarize the content of the 11^{th} lesson of tutoring on the course Rings and modules. Also, attached in the end, there is an exercise sheet.

Contents

1. Class: Noetherian modules

In one of our beginning lessons we stressed the importance of being kind of finite dimensional. Today we will focus on a smallness notion that is module theory had much success and was deeply studied: noetherianity. The strcuture of this lesson is the following. Fist we see a short theoretical digression, then we solve some exercises.

1.1. Generalities on noetherianity.

Definition 1. A modules is noetherian iff any submodule is finitely generated. Equivalently, if any incresing chain

$$N_0 \subset N_1 \subset \dots$$

of submodules is eventually stationary.

There are many natural examples of noetherian modues over a ring.

$\mathbb{Z}$	✓
$\mathbb{K}$	✓
$\mathbb{K}[x]$	✓
$\mathbb{K}[x, y]$	✓
$\mathbb{Z}[x]$	✓

But not every ring is noetherian.

$\prod_1^\infty \mathbb{Z}$	✗
$\sum_1^\infty \mathbb{Z}$	✗
$\mathbb{K}[x_1 \dots]$	✗

The idea behind noetherianity is that as soon as we are working with classical algebraic geometry (i.e. varieties of family of polynomials over a field), everything is noetherian.

Remark 2. Indeed a noetherian module is finitely generated.

Definition 3. A ring which is a noetherian module over itself is called noetherian.

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We are used to the fact that smallness conditions change rigidly the behaviour of a module. The folloing proposition an example of this phenomenology.

Proposition 1.1. Consider an exact sequence of modules.

$$0 \to N \to M \xrightarrow{p} P \to 0.$$

The following are equivalent:

(1) M is noetherian.
(2) N and P are noetherian.

Proof. Assume (1). Since N is a submodule of M, it is noetherian. Now observe that a family of submodules P_i of P corresponds to a family of submodules $p^{-1}(P_i)$. of M. This sequence must be stationary because M is noetherian, so their projection is stationary. This proves that P is noetherian too.
Now assume (2). Consider an ascending chain M_i. We want to prove that it is stationary. If we look at the inclusion of M_i into M:

$$\begin{array}{ccccccccc} & & N_i & \longrightarrow & M_i & & & & \\ & & \downarrow & & \downarrow & & & & \\ 0 & \longrightarrow & N & \longrightarrow & M & \longrightarrow & P & \xrightarrow{p} & 0 \end{array}$$

We can consider the pullback N_i as in the diagram above. Call P_i the coequalizer of this pullback. Observe that since N_i is in the kernel of p, there is a map $P_i \to P$.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N_i & \longrightarrow & M_i & \longrightarrow & P_i & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \vdots & & \\ 0 & \longrightarrow & N & \longrightarrow & M & \longrightarrow & P & \xrightarrow{p} & 0 \end{array}$$

And when $i < j$ we get a diagram like the following one.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N_i & \longrightarrow & M_i & \longrightarrow & P_i & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \vdots & & \\ 0 & \longrightarrow & N_j & \longrightarrow & M_j & \longrightarrow & P_j & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \vdots & & \\ 0 & \longrightarrow & N & \longrightarrow & M & \longrightarrow & P & \xrightarrow{p} & 0 \end{array}$$

Now observe that eventually in j, the maps $N_i \to N_j$ and $P_i \to P_j$ are isomorphisms, because N and P are noetherian. By 5 lemma we conclude that eventually M_i is isomorphic to M_j, which is the thesis. □

1.2. **Bestiary.** In this subsection we see some recurrent ideas when dealing with noetherian rings.

Exercise 1. Let A be a noetherian ring and ϕ a surjective morphism $A \to A$. Then ϕ is injective.

Proof. Consider the following chain of submodules

$$\mathrm{Ker}(\phi) \subset \mathrm{Ker}(\phi^2) \subset \cdots$$

This sequence must be stationary. So there exists n such that $\mathrm{Ker}(\phi^n) = \mathrm{Ker}(\phi^{n+1})$. Now consider an element $a \in \mathrm{Ker}(\phi)$. We want to prove that a is 0. Since ϕ is surjective, so is ϕ^n. Thus there is an element b such that $a = \phi^n(b)$. Clearly $\phi(a) = \phi^{n+1}(b) = 0$. But since $\mathrm{Ker}(\phi^n) = \mathrm{Ker}(\phi^{n+1})$, we have that $\phi^n(b) = 0$, thus

$$a = \phi^n(b) = 0.$$

□

Definition 4. Recall that a local ring A is a ring with just one maximal ideal, generally called $\mathfrak{m}$.

The following exercise tell us that some properties are hereditary in noetherian rings.

Exercise 2. A local noetherian ring A such that $\mathfrak{m}$ is principal has principal ideals.

Proof. Consider an ideal $I \subset \mathfrak{m} = (m)$. The idea is quite easy. I must be finitely generated$I = (i_1 \dots i_n)$. We will prove that each $i_n = a_n m^{k_n}$. where a_n is invertible. This is equivalent to the thesis. Now consider i_n. Since $(i_n) \subset (m)$, m must divide i_n, $m|i_n$, i.e. there exist a_1 such that $i_n = a_1 m$. Moreover $(i_n) \subset (a_1)$. We can repeat this argument for a_1 building a chain $(i_n) \subset (a_1) \subset \dots$ which must be stationary. So after a finite number of steps a_k is invertible and thus we get

$$i_n = a_n m^{k_n}.$$

□

2. Exercises

Pay attention, exercises labelled by the tea cup ☕ may not be incredibly challenging, even not challenging, but it is important keep them in mind, so take your time when solving them and be careful to find a formal and correct solution. Exercises labelled by the danger international sign ⚠ are very challenging.