RINGS AND MODULES

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This note is going to summarize the content of the second lesson of tutoring on the course Rings and modules. Also, attached in the end, there is an exercise sheet.

Contents

1. Class

Today class will have two parts: computations and new ideas. The first section is intended to familiarise with modules in a very concrete way; the second wants to give an overview on the zoology that one can encounter when studying modules. This very wide zoology has some rigidities which is very important to point out.

Remark 1. Remember, rings are assumed to be commutative and with identity when needed.

1.1. Computations.

Exercise 1. Compute $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z})$.

Proof. The mapping $f \mapsto f(1)$ is an iso between $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z})$ and $\mathbb{Z}$. □

Exercise 2. Compute $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Z})$.

Proof. Consider an element $f \in \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Z})$ and just follow the line above.

$$f\left(\frac{a}{b}\right) = f\left(\frac{pa}{pb}\right) = p \cdot f\left(\frac{a}{pb}\right).$$

Thus, if we suppose $f(\frac{a}{b}) \neq 0$, then any prime p divides $f(\frac{a}{b})$, but the only number divided by any prime is 0. This is just absurd, and there are no elements in $f \in \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Z})$. □

Exercise 3. Compute $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}_n, \mathbb{Z}_m)$.

Proof. We give a combinatorial presentation of its cardinality. An homomorphism f is completely determined by $f(1)$. Thus it is enough to understand how many possibilities we have for $f(1)$. Since $n \cdot 1 = 0$ we have that $n \cdot f(1) = 0$. Thus the question becomes, how many solution has the equation

$$nx \equiv 0 \qquad (m)?$$

□

Date: 5 October 2017.

Exercise 4. Let R be a ring and e be an element such that $e^2 = e$. Prove the following:

(1) $e \cdot A$ is a submodule of A,
(2) $e \cdot A$ is a retract of A,
(3) $A \cong (1-e) \cdot A \oplus e \cdot A$.

Proof. (1) This is just trivial.
(2) Consider the map $a \mapsto ea$.
(3) Consider the map $((1-e)a, eb) \mapsto (1-e)a + eb$. This is clearly surjective because $x = (1-e) \cdot x - e \cdot x$. Moreover it is injective. Suppose the contrary, then we have $(1-e)a = eb$ for some $a, b \in A$. But this is absurd because $0 = e(1-e)a = e^2 b = eb \neq 0$.

□

1.2. **The importance of being finite.** It was very important to make some explicit computations because at some point of your life someone will ask you if you suffered enough; now you have a very affirmative answer. Here we come to some fun. The slogan of this section is the following statement:

$$A^n \ncong A^m.$$

Although it seems quite natural and surely is absolutely trivial for vector spaces, it's quite a theorem in module theory. I purpose you this result and all others of the section because I want to show that some natural claims are true, but the are absolutely not evident.

Definition 2. A module M is finitely generated if there is a surjective morphism $A^n \overset{f}{\twoheadrightarrow} M$.

Remark 3. This is equivalent to require the existence of a finite generator, i.e. a set $\{m_i\}$ such that any element m is a linear combinations of elements of the set. $m = \sum a_i m_i$. Let's prove it.

Proof. Consider an element m. Since the map is surjective there is an element $a \in A^n$ such that $f(a) = m$. Now, A^n has a natural set $\{e_i\}$ of generators such that $a = \sum a_i e_i$, so $m = f(a) = \sum a_i f(e_i)$. This proves that the set $\{f(e_i)\}$ is a finite set of generators for M. □

Finitely generated modules, have many rigidity property, as their cousins finite dimensional vector spaces have. We will start from a very important result that we will not prove today.

Theorem 1.1 (Nakayama)**.** Let M be a finitely generated module and I an ideal such that $I \cdot M = M$. Then there is an element $i \in I$ such that $i \cdot m = m$ for all $m \in M$.

Is this a powerful result? Sure it is! We can conclude many rigidity results of finitely generated modules from this. The last one will be our slogan.

Theorem 1.2. Every surjective endomorphism $M \xrightarrow{f} M$ of a finitely generated module is injective.

Proof. As we have seen M has a natural $A[x]$-module structure such that

$$p \cdot m = p(f)(m).$$

Since f is surjective

$$(x) \cdot M = f(M) = M.$$

By Nakayama lemma there is an element $q \in (x)$ such that $q \cdot m = m$, equivalently $(1 - q) \cdot m = 0$. Now consider $m \in \mathrm{Ker}(f)$.

$$\begin{aligned} m &= 1 \cdot m \\ &= (1 - q + q) \cdot m \\ &= (1 - q) \cdot m + q \cdot m \\ &= 0 + q(f)(m) \\ &= 0 + 0 \\ &= 0. \end{aligned}$$

□

The following result proves that a finitely generated module cannot be isomorphic to a proper quotient of itself.

Theorem 1.3. Let M be a finitely generated module and $f : M \twoheadrightarrow N$ a surjective morphism which has a non-trivial kernel. Then there cannot be an iso $\phi : N \to M$.

Proof. Suppose there is an isomorphism $\phi : M \to N$, then the composition

$$M \xrightarrow{f} N \xrightarrow{\phi} M,$$

is a surjective endomorphism of M, thus it must be an isomorphism. But f has a non-trivial kernel! □

Corollary 1.4. $A^n \ncong A^m$

Proof. If $m < n$, A^m is just a proper quotient of A^n. □

2. Exercises

Pay attention, exercises labelled by the tea cup ☕ may not be incredibly challenging, even not challenging, but it is important keep them in mind, so take your time when solving them and be careful to find a formal and correct solution. Exercises labelled by the danger international sign ⚠ are very challenging.

Exercise 5. Find a counterexample to Nakayama lemma when M is not finitely generated.

Exercise 6. ☕ Find a counterexample to Theorem 1.3 when M is not finitely generated.

Exercise 7. Show that $\mathbb{R}$ is not a finitely generated $\mathbb{Q}$-module.

Exercise 8. Show that $\mathbb{Q}$ is not a finitely generated $\mathbb{Z}$-module.

Exercise 9. Finish Exercise 3.

Exercise 10. Prove that $\mathrm{Hom}_A(A, M) \cong M$.

Exercise 11. ⚠ Nakayama Lemma is just false in non commutative rings. Can you find a counterexample?