RINGS AND MODULES

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This note is going to summarize the content of the third lesson of tutoring on the course Rings and modules. Also, attached in the end, there is an exercise sheet.

Contents

1. Class: Trace operators

Today we solve a problem. You have seen many wonderful tools of module theory, such as projectives and tensor product. I would like to show you some applications of these ideas to a chosen problem. Results we see today are not something you must remember, nor something you will use in your life but techniques and ideas we are going to use as tool when solving the problem are standard tools in module theory. We are learning how to fish, by going fishing.

A well know result in linear algebra is the following theorem.

Theorem 1.1. Consider the vector space of matrices $\mathrm{M}(\mathbb{K}, n)$ and a linear operator

$$t : \mathrm{M}(\mathbb{K}, n) \to \mathbb{K}$$

such that $t(AB) = t(BA)$. Then $t = \lambda \mathrm{tr}$ for a suitable λ in $\mathbb{K}$.

Since in vector space every module is free we can rephrase in a more fancy way.

Theorem 1.2. Consider the vector space $\mathrm{End}(V)$ for a finitely generated vector space V, and a linear operator

$$t : \mathrm{End}(V) \to \mathbb{K}$$

such that $t(f \circ g) = t(g \circ f)$. Then $t = \lambda \mathrm{tr}$ for a suitable λ in $\mathbb{K}$.

This theorem says that operators with the property $t(AB) = t(BA)$ are essentially unique in vector spaces. Equivalently, the subspace $\mathrm{Tr} \subset \mathrm{Hom}(\mathrm{End}(V), \mathbb{K})$ of operators such that $t(AB) = t(BA)$ has dimension 1. So here comes a natural question, is this still true in modules?

There are many problems when asking this question.

- We have not anymore a candidate solution for the generator of Tr, since what is the trace of an endomorphism of a module?
- We have no notion of dimension.

Although we have no clear formulation, what we want is quite clear. How many, essentially different, elements are there inside Tr?

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Problem 1.3. Understand the submodule $\mathrm{Tr} \subset \mathrm{Hom}(\mathrm{End}(M), A)$ of operators such that $t(f \circ g) = t(g \circ f)$ for a given finitely generated module M. We call an element $t \in \mathrm{Tr}$ a trace operator.

Let's start with a concrete example.

Exercise 1. How many trace operators are there when $M = \mathbb{Z}_6$ and $A = \mathbb{Z}$?

Proof. And in fact there are no trace operators but 0. First, observe that for any $f \in \mathrm{End}(\mathbb{Z}_6)$ we have that $6 \cdot f$ is the 0 function. In fact,

$$(6 \cdot f)(v) = f(6 \cdot v) = f(0) = 0.$$

Now suppose there is one and call it t. Call f and element such that $t(f) \neq 0$. Here comes the absurd.

$$0 \neq 6 \cdot t(f) = t(6 \cdot f) = t(0) = 0.$$

□

Now we come to a case in which we can recover in some sense hypotheses of we lost: basis and trace.

1.1. **The free case.** The case $M = A^n$ behaves more like vector spaces.

Theorem 1.4. If $M = A^n$ and A is domain, a trace operator is a normalization of the sum of elements on the diagonal of the matrix.

Proof. When A is a domain one can tensor by quotients $\mathbb{Q}(A)$ of A and get an inclusion,

$$\mathrm{End(M)} \subset \mathrm{End}(M) \otimes_A \mathbb{Q}(A).$$

If one can extend the operator to the whole module, then the answer is close... So suppose to have an operator $tr : \mathrm{End(M)} \to A$, because of linearity it's enough to define $\bar{tr}$ on $E_{ij} \otimes \frac{a}{b}$. We put

$$\bar{tr}(E_{ij} \otimes \frac{a}{b}) = tr(E_{ij}) \otimes \frac{a}{b}.$$

One can see this map as the map $tr \otimes \mathrm{id}$, obtainted by functoriality of tensor product. This extends the operator to a linear operator

$$\mathrm{End}(M) \otimes_A \mathbb{Q}(A) \xrightarrow{\bar{tr}} A \otimes_A \mathbb{Q}(A) = \mathbb{Q}(A).$$

Is $\bar{tr}$ a trace operator? We can prove it again on generators.

$$\begin{aligned}
\bar{tr}(E_{ij} \otimes \frac{a}{b} \circ E_{hk} \otimes \frac{c}{d}) &= \bar{tr}(E_{ij} \circ E_{hk} \otimes \frac{ac}{bd}) \\
&= tr(E_{hk} \circ E_{ij}) \otimes \frac{ac}{bd} \\
&= tr(E_{hk} \circ E_{ij}) \otimes \frac{ac}{bd} \\
&= \bar{tr}(E_{hk} \otimes \frac{c}{d} \circ E_{ji} \otimes \frac{a}{b}).
\end{aligned}$$

We proved that any trace operator on A^n is a restriction of a trace operator on $\mathbb{Q}(A)^n$, thus we can apply the classification of trace operator on vector spaces. □

Remark 1. We strongly used the hypotesis that to tensor is not going to kill anything. So $\mathrm{Ann}(m)$ shoud be 0 for any element to make this proof work. In the case $M = \mathbb{Z}_6$ we have $\mathrm{End(M)} \not\subset \mathrm{End}(M) \otimes_A \mathbb{Q}(A)$.

Are we happy of this result? Is this enough? Well, this is a good result but it is quite unsatisfactory. In the first lesson we have seen that many modules behave very differently from A^n, so we cannot guess what is going to happen for a generic module.

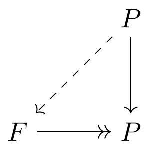
We want now to study this situation when M is projective. Thus we are going to call it P. We will start with some general observation on projective modules.

1.2. **The projective case.**

Remark 2. Any projective, finitely generated module is covered by a free module, as we know from first lesson,

$$F \overset{p}{\twoheadrightarrow} P.$$

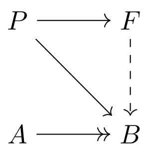
Projective modules have a very special property, this surjection splits. This is quite easy to prove. Let's give a look to the following diagram.



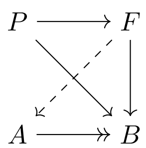
Where the dotted arrow exists because P is projective. This map must be injective, because the composition with the surjection is the identity. Thus F splits

$$F = P \oplus \mathrm{Ker}(p).$$

Remark 3. When P is a direct summand of F, P is projective. This is quite easy to prove too.



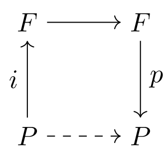
Since P is a direct summand, we can extend any map $P \to B$ to a map $F \to B$, by putting the map 0 elsewhere.



Now we can find a map $F \to A$ because F is projective, and the composition $P \to F \to A$ is the map we were looking for.

Corollary 1.5. Projective modules are precisely direct summands of free ones.

Remark 4. If we have a projective module P and a cover for it $F \twoheadrightarrow P$. We can push endomorphism of F to endomorphisms of P. We call p the projection and i the section $P \to F$.



So we get a projection

$$\mathrm{End}(F) \twoheadrightarrow \mathrm{End}(P),$$

mapping $f \mapsto pfi$. In fact, as before, one can prove that $\mathrm{End}(P)$ is a direct summand of $\mathrm{End}(F)$.

Remark 5. In linear algebra, for any finitely generated vector space V, one has that the dual

$$\mathrm{Hom}(V, \mathbb{K}) \cong V.$$

This is totally false in module theory, the easy counterexample is our dear $\mathbb{Z}$-module $\mathbb{Z}_6$. Nonetheless, if P is finitely generated projective module, one gets back the statement

$$\mathrm{Hom}(P, A) \cong P.$$

Proof. The proof is very instructive but quite tricky. First remember that there is a free module F such that $F = P \oplus N$, where N is projective too. If P is finitely generated we can choose F to be A^n for a suitable n. Surely we have that

$$A^n \cong \mathrm{Hom}(A, A^n) \cong \bigoplus_1^n \mathrm{Hom}(A, A) \cong \mathrm{Hom}(A^n, A).$$

In this equation we used extensively that modules have finite biproducts. Now if we substitute A^n with $P \oplus N$ we get:

$$P \oplus N \cong \mathrm{Hom}(P, A) \oplus \mathrm{Hom}(N, A).$$

In principle this is not enough to prove the thesis, but if one follows precisely the chain of isos discovers that P has image in $\mathrm{Hom}(P, A)$. □

Remark 6. Now comes some magic. Be prepared. We shall tell where we want to get, and the prove it. We want to prove that when P is projective there is at least a trace operator

$$t : \mathrm{End}(P) \to A.$$

But how to do so? The idea is that we can find a natural map

$$t : \mathrm{Hom}(P, A) \otimes P \to A,$$

which maps $\sum a_i f_i \otimes p_i \mapsto \sum a_i f_i(p_i)$ and observe that for a projective module

$$\mathrm{Hom}(P, A) \otimes P \cong \mathrm{End}(P).$$

The obtained composition is a trace operator.

Remark 7. $\mathrm{Hom}(P, A) \otimes P \cong \mathrm{End}(P)$.

Proof. By abstract nonsense we can prove that

$$\mathrm{Hom}(\mathrm{Hom}(P, A) \otimes P, A) \cong \mathrm{Hom}(\mathrm{Hom}(P, A), \mathrm{Hom}(P, A)) \cong \mathrm{Hom}(P, P).$$

If we could prove that $\mathrm{Hom}(P, A) \otimes P$ is projective, than we are done. Since $\mathrm{Hom}(P, A) \cong P$ it is enough to prove that $P \otimes P$ is projective. This is a good exercise for you, and you find it in the exercise sheet. □

Theorem 1.6. Let P be a projective module on a domain A. Then there is precisely 1 trace operator up to normalization.

Proof. Consider a trace operator

$$t : \mathrm{End}(P) \to A.$$

Remember that the finite cover $F \to P$ splits, so $F \cong P \oplus N$ and N is projective too, so there is an other trace operator s. We call i, p the inclusion and the projection for P and j, n the inclusion and the projection for N. It might sound surprising, but the operator

$$\bar{t} : \mathrm{End}(F) \to A.$$

defined by $\bar{t}(fg) = at(pfgi) + bs(nfgj)$ is a trace operator for a suitable choice of $a, b \in A$.

Thus we proved that a trace operator on P is just the restriction of a trace operator on F, but we studied this scenario before.

$\square$

Remark 8. It is quite natural to ask how the hell one should find the $\bar{t}$ we presented in last theorem. Especially, what about a and b? The idea is in fact quite simple. If we have a matrix

$$M = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$$

Its trace is simply the same of the matrix

$$M = \left[\begin{array}{c|c} A & 0 \\ \hline 0 & D \end{array}\right]$$

A and B are respectively pMi and nMj. So that $tr(M) = tr(A) + tr(B)$. Now, if you have two traces defined on the subspaces indicated by the two squares A and B, they are a restriction up to scalar of the trace defined on the environment, this is where a and b appear.

Remark 9. $\mathbb{Z}_6$ is not a projective $\mathbb{Z}$-module.

Problem 1.7. We come againg to our question, what about a generic M? Honestly, I don't know. The problem of trace operators have been formulated with the purpose of keeping this lesson, there is no literature on the subject and I have not investigated all possibilities, but you could try! You have tools!

Remark 10. So, what we learned today? Nothing, I would say. We learned that we know enough theory to destroy a problem. We discovered that we already knew. This is learning, Plato.

2. Exercises

Pay attention, exercises labelled by the tea cup ☕ may not be incredibly challenging, even not challenging, but it is important keep them in mind, so take your time when solving them and be careful to find a formal and correct solution. Exercises labelled by the danger international sign ⚠ are very challenging.

Exercise 2. ☕ Read today class twice. There are so many tricks to learn.

Exercise 3. Consider two modules M, N. Prove that

$$\mathrm{End}(M \oplus N) \cong \mathrm{End}(M) \oplus \mathrm{End}(N) \oplus \mathrm{Hom}(M, N) \oplus \mathrm{Hom}(N, M),$$

then use this result to conclude that for any couple of integers a, b one has that $(a+b)^2 = a^2 + b^2 + 2ab$.

Exercise 4. In Remark 6 I did not prove that the obtained operator is a trace operator. Prove it.

Exercise 5. If P, Q are projective finitely generated modules, than $P \otimes Q$ is projective too.

Exercise 6. In Theorem 1.6 I did not prove that $\bar{t}$ is a trace operator. Prove it. At some point you might need that $\mathrm{id}_F = jn + ip$.

Exercise 7. ⚠ This is addressed to the wannabe category theorist. What can you say about trace operators when M is $\mathbb{R}[x]$ and $A = \mathbb{R}$?