RINGS AND MODULES

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This note is going to summarize the content of the 4^{rd} lesson of tutoring on the course Rings and modules. Also, attached in the end, there is an exercise sheet.

Contents

1. Class

Today class will have two parts. In the last lesson we have seen that sometimes projectives behave like free modules. In the first part of the lesson we will that some strong assumptions on the ring will turn any projective into free modules. In the second part we will look at two notions of smallness for modules.

1.1. PID have trivial projectives.

Definition 1. A ring A is PID if it is a domain and every ideal I is generated by an element $I = (a)$.

$\mathbb{Z}$	✓
$\mathbb{K}$	✓
$\mathbb{K}[x]$	✓
$\mathbb{K}[x, y]$	✗
$\mathbb{Z}[x]$	✗

Remark 2. Recall that any euclidean domain is PID and that any PID is UFD.

Theorem 1.1. For PID rings, any submodule of a free module is free.

Proof. We will prove the theorem just when F is finitely generated. The proof can be generalized easly, but I cannot assume that you have familiarity with ordinals. Consider a basis $\{e_i\}$ of F. We will prove by induction on the cardinality of the basis that any submodule is free. In the case that $F = A$, the condition that any submodule is free is precisely the condition that A is PID. The hypothesis of domain is used because $(a) \cong A/\text{Ann}(a)$. Now let's focus on the inductive step. Call M the submodule. Since the basis of F has cardinality $n + 1$ we can split $F = F_n \oplus < e_{n+1} >$. Call $M_n = M \cap F_n$. If $M_n = M$, we are just done by inductive hypotesis. Suppose now it is not the case, now for each element $m \in M$ we have that

$$m = m_n + a \cdot m_{n+1}.$$

Define I to be the ideal $\{a \in A : \exists m_n \in M_n : a \cdot e_{n+1} + m_n \in M\}$. This is an ideal and must be principal, so let's say $I = (a)$. Now one can show that $M = M_n \oplus < am_{n+1} >$, and we are done. □

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Corollary 1.2. Projectives are free, when A is a PID.

1.2. **Two notion of smallness.** Now we will give a look on two different notions of smallness for a module. One is a tribute to this department, the other was very important in the history and the characterization of abelian categories. They will look quite similar to you.

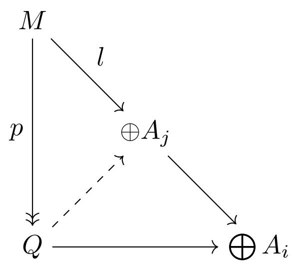
1.2.1. $\oplus$-*smallness.*

Exercise 1. A module M is $\oplus$-small if every morphism $f : M \to \oplus A_i$ factors through a finite direct sum of A_i's.

Let M be a $\oplus$-small objects. Prove that the following:

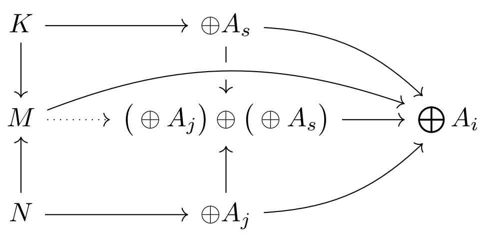
(1) Quotients of M are $\oplus$-small.
(2) $M = N \oplus K$ is $\oplus$-small iff N, K are $\oplus$-small.

Proof. (1) Just look at the diagram,



where the dotted arrow exists because l must vanish on the kernel of p.

(2) Because of (1) N and K must be $\oplus$-small, they are quotients of M. When N and K are $\oplus$-small, we can look at the following diagram:



Where the dotted arrow exists because M has the universal property of the sum of K and N. Since the union of finite sets if finite, the dotted map is a solution for our problem. □

Now comes a natural question. Which modules are $\oplus$-small? Here we try to give a partial answer. We will discover that finitely generated modules are $\oplus$-small.

Exercise 2. A finitely generated free module is $\oplus$-small.

Proof. This is very easy. □

Corollary 1.3. A finitely generated module is $\oplus$-small.

Proof. It is a quotients of a finitely generated free one. □

1.2.2. *ω-smallness.* The notion of ω-smallness is related to nowadays research work at MU. The idea is the following. The is a notion of smallness very similar to the one we have seen before which is totally categorical and characterized finitely generated modules. Thus this notion can be moved to any category and be employed to study small objects in other context.

The only theorem that we will prove in this section is that a modules if ω-small if and only if is finitely generated. Is this a failure for this definition just because we rediscovered something we already knew? No, contrariwise this is a bridge to give a notion of finitely generated object in any category!

Definition 3. A direct system of modules $\{M_i\}$ is a class of modules such that given two modules M_i and M_j there is a module M_k is the system such that M_i and M_j are submodules of M_k.

Remark 4. The union of all modules in a directed system is a module.

Definition 5. Let M_i be a chain of modules. A module M is ω-small if every morphism $f : M \to \bigcup M_i$ where the family of $\{M_i\}$ is a directed system factors through an M_i.

Exercise 3. Quotients of ω-small modules are ω-small.

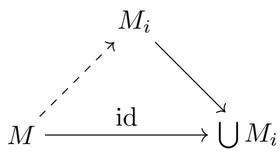
Proof. This is just the same of 1. □

Exercise 4. A finitely generated free module is ω-small.

Proof. Let $\{e_i\}$ be a basis for F. Call M_i the module generated by $< f(e_i) >$. By induction you find an M_t where all this modules sit on. That is the solution. □

Exercise 5. ω-small modules are precisely finitely generated modules.

Proof. Exercises 3 and 4 prove together that finitely generated modules are ω-small. On the other hand, consider an ω-small module M. Now observe that a module is the union of the directed system of its finitely generated submodules. and look at the following diagram:



And since M_i is finitely generated, so must be M.

□

2. Exercises

Pay attention, exercises labelled by the tea cup ☕ may not be incredibly challenging, even not challenging, but it is important keep them in mind, so take your time when solving them and be careful to find a formal and correct solution. Exercises labelled by the danger international sign ⚠ are very challenging.

Exercise 6. ☕ Enjoy your tea.