RINGS AND MODULES

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This note is going to summarize the content of the 5^{rd} lesson of tutoring on the course Rings and modules. Also, attached in the end, there is an exercise sheet.

Contents

1. Class

Today we focus on the topic of flat modules. nothing special today, we will give a close look to some properties of flat modules to familiarize with the subject.

This list of properties has no importance, the aim of the lesson is to get used to the most standard techniques. In this sense, proofs are much more important than statements today.

Exercise 1. Let A be a commutative ring. Prove that the following are equivalent:
(1) $a \in a^2A$.
(2) aA is a direct summand of A
(3) A/aA is flat.

Proof. Assume (1). Thus, $a = a^2b$ for some b. To prove (2) It is enough to find a section $aA \to A$. This map is precisely $am \mapsto b \cdot am$. Now assume (2). Since A is free, it is projective. A direct summand of a projective module is projective and thus flat. Now assume (3). Give a look to the following exact sequence:

$$0 \to aA \to A \to A/aA \to 0.$$

Since we assume A/aA to be flat we can tensor and obtain an other exact sequence.

$$0 \to aA \otimes A/aA \to A \otimes A/aA \to A/aA \otimes A/aA \to 0.$$

This sequence simplifies in the following:

$$0 \to aA/a^2A \to A/aA \to A/aA \to 0.$$

Since the second map is the identity, it must be the case that:

$$aA/a^2A \cong 0.$$

This proves (1), and we are done. □

Remark 1. When we have a module M, we can rethink $M/I \cdot M$ as the tensor product $M \otimes A/I$.

Proof. Consider the correspondence $M \to M \otimes A/I$ mapping $m \mapsto m \otimes 1$. This map is clearly surjective and an element in the kernel is precisely and element of $I \cdot M$. □

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Exercise 2. When M is flat, then $I \cdot M \cong M \otimes I$.

Proof. Consider the exact sequence

$$0 \to I \to A \to A/I \to 0$$

and tensor by M. □

Exercise 3. Let M be a flat module and $a \in A$ an element which is not a 0 divisor. Then $a \cdot m \neq 0$ for all $m \in M$.

Proof. Consider the exact sequence

$$0 \to 0 \to A \to aA \to 0,$$

then just tensor by M. □

Exercise 4. Let M be a module such that $m \cdot M \neq M$ for all maximal ideal $m \subset A$. Prove the following.

(1) for all I, proper ideals of A we have that $M/IM \neq 0$.
(2) If M is flat, then for all N, $M \otimes N \neq 0$.

Proof. (1) Since I is proper there is a maximal ideal m that contains I. So $IM \subset mM \neq M$.
(2) Consider an element $n \in N$. We can point at this element with a map

$$0 \to A/\mathrm{Ann}(n) \to N.$$

Because of flatness of M we can tensor and get an injective map

$$0 \to A/\mathrm{Ann}(n) \otimes M \to N \otimes M.$$

We can rewrite this sequence as follows:

$$0 \to M/\mathrm{Ann}(n)M \otimes M \to N \otimes M.$$

By (1), $M/\mathrm{Ann}(n)M$ is not 0.

□

2. Exercises

Pay attention, exercises labelled by the tea cup ☕ may not be incredibly challenging, even not challenging, but it is important keep them in mind, so take your time when solving them and be careful to find a formal and correct solution. Exercises labelled by the danger international sign ⚠ are very challenging.

Exercise 5. Enjoy your ☕.