RINGS AND MODULES

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This note is going to summarize the content of the 6^{th} lesson of tutoring on the course Rings and modules. Also, attached in the end, there is an exercise sheet.

CONTENTS

1. CLASS: ABELIAN CATEGORIES

Today we are going to give a categorical insight on the theory of modules. We shall start by an analysis of a category of modules.

Let A be a commutative ring and $\text{Mod}(A)$ the category of modules over A. Of course $\text{Mod}(A)$ has many properties we have already seen on these blackboards.

(1) There is a terminal and an initial object and in fact these two objects coincides;
(2) There are kernels and cokernels;
(3) There are finite products and sums;
(4) Every mono is the kernel of a map;
(5) Every epi is the cokernel of a map.

These structural properties are not everywhere around, the category of topological spaces fails to verify (1), as well as the category of sets.

Today we would argue that any category verifying this list is in fact the category of modules of a ring. In fact, we will restring to a simpler case to show a simplified argument.

Although this is the goal of today, we will start by recovering some properties of abelian categories. For notational convenience, today, $\mathscr{A}$ is always an abelian category.

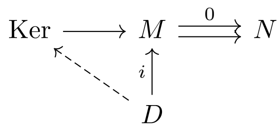
1.1. **Familiarize.**

Remark 1. Given two objects M, N in an abelian category there is always a map between them, called the 0 map. It's the composition of $M \to 0 \to N$. Recall that this is uniquely identified by the fact that the terminal object is initial.

Proposition 1.1. The kernel of a mono is 0.

Proof. Let $M \xrightarrow{f} N$ be a mono. Recall that the kernel has the universal property identified by this diagram:

Date: 2 November 2017.



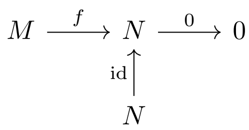
Now, we prove that $0 \to M$ has the desired property. Suppose $fi = 0$, then $fi = 0 = f0$. Since f is monic, i is forced to be 0, and $D = 0 \to 0$ is the desired factorization of i. □

Remark 2. Dually, the cokernel of an epi $f : M \to N$ is the 0 map.

Fact 1. f is the kernel of its cokernel.

Proposition 1.2. In an abelian category $\mathscr{A}$, any map which is both an epi and a mono is an iso.

Proof. Let $f : M \to N$ be both an epi and a mono. The coker of f is $N \to 0$ because f is an epi. The kernel of the 0 map is the identity of N.



Because of fact 1 also M is this kernel, thus M and N must be isomorphic. We this not prove that f is an iso, but this is already encoded in the diagram. □

Remark 3. Recall, this is not trivial in a generic category.

- category of sets has this property
- the category of topological spaces does not have this property.

1.2. **Embedding.**

Definition 4. An object P in a category $\mathscr{K}$ is a generator if it can distinguish maps, i.e. given two different parallel arrows $M \rightrightarrows N$ there exists a map $P \to M$ such that the two compositions $P \to M \rightrightarrows N$ are still different.

Remark 5. This is a toy notion of generator. There are much more interesting ones, today we stack on this one not to complicate things.

- $\{\bullet\}$ is a generator in the category of sets;
- $\{\bullet\}$ is a generator in the category of topological spaces;
- $\mathbb{Z}$ is a generator in the category of abelian groups.

Proposition 1.3. In the category $\mathrm{Mod}(A)$, A is a generator.

Proof. Suppose we have two different maps $M \rightrightarrows N$, then there is an element $m \in M$ on which they act differently. Now consider the map $A \xrightarrow{m} M$ mapping $1 \mapsto m$. This is the desired point. □

Theorem 1.4. Let $\mathscr{K}$ be a category with a generator P, then the functor:

$$\mathscr{K} \xrightarrow{\hom(P,\_)} \mathrm{Set},$$

is faithful.

Proof. A functor F is faithful precisely when $f \neq g$ implies $F(f) \neq F(g)$. So, consider two different arrows $f, g : M \to N$, since P is a generator there is a map $P \xrightarrow{p} M$ that distinguishes them. Now $\hom(P, f)(p) = f \circ p$, and $\hom(P, g)(p) = g \circ p$. Thus they differ on an element, namely p. □

Now, give an abelian category $\mathscr{A}$ we can build a set, this looks quite far from being a module on a ring. Especially, we need a candidate ring on which building the module structure.

Proposition 1.5. $\hom(P, M)$ has a natural $\mathrm{End}(P)$-module structure.

Proof. This is very easy for an endomorphism ϕ and a point $g \in \hom(P, M)$ we define

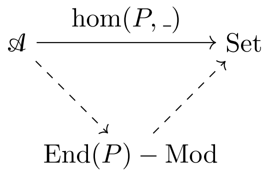
$$\phi \cdot g = g \circ \phi.$$

This is a right module structure on $\hom(P, M)$. □

Remark 6. Hey, I totally tricked you. For sure $\hom(P, M)$ is a monoid, but who proved it is a ring? I totally omitted a very important general fact, an abelian category has a natural structure of abelian group on each $\mathrm{Hom}(M, N)$. This is a totally untrivial result that proves that some natural and apparently unbiased hypotheses on the category are going to generate so much structure. An exercise could guide you on setting this structure but it is better to give you two solid references in which looking for this proof. One is the volume two of Borceux, *Handbook of categorical Algebra*, the other is *Abelian Categories* by Freyd.

Theorem 1.6. An ebelian categoty with a generator is always a subcategory of a category of modules.

Proof. Not much to prove here, them hom functor factors through the category of $\mathrm{End}(P)$ modules as proved by the proposition above.



□

Remark 7. Observe that P is projective if and only if this functor preserves exact sequences, so if the abelian category has a projective object, one can sharpen this result providing a functor which preserves much more structure, even exactness properties of the category.

2. Exercises

Pay attention, exercises labelled by the tea cup ☕ may not be incredibly challenging, even not challenging, but it is important keep them in mind, so take your time when solving them and be careful to find a formal and correct solution. Exercises labelled by the danger international sign ⚠ are very challenging.

Exercise 1. Prove the Remark 7.