RINGS AND MODULES

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This note is going to summarize the content of the 7^{th} lesson of tutoring on the course Rings and modules. Also, attached in the end, there is an exercise sheet.

Contents

1. Class: Injective modules

Finally, today we see something about injectives. I shall start from a general discussion of the subject. This discussion can be found on the beautiful book *Algebra Chapter 0*, by Aluffi.

1.1. General discussion.

We may have inadvertently communicated to my reader the message that every categorical notion leads in a straightforward way to a mirror notion, by 'just reversing arrows'. This would seem to be the case for injective vs. projective modules: the definition of one kind is indeed obtained from the definition of the other by simply reversing arrows in the key diagrams.

However, this is as far as this naive perception can go. A seemingly unavoidable asymmetry of nature manifests itself here, making injective modules look more mysterious than projectives.

Remark 1. For example, there is no 'easy' description of injective modules in the style of:

Proposition 1.1. *P is projective if and only if it is a direct summand of a free module.*

Remark 2. Moreover, while it is trivially the case that an arbitrary module is surjected upon by a projective (e.g., free) module, the 'mirror' statement for injectives is certainly not trivial.

Remark 3. The problem is that it is a little hard to picture an injective module: while A itself is a trivial example of a projective A-module, at this point the reader would likely find it hard to name a single nonzero injective module over any ring whatsoever. Well, say over any ring that is not a field: if A is a field, then A itself is injective. But if$A$ is not a field? Suppose r is a non-zero-divisor in A, and consider the inclusion of the ideal $(r) \hookrightarrow A$:

$$0 \to A \stackrel{\cdot r}{\to} A :$$

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extending the identity on$A$ through the multiplication by r:

$$\begin{array}{ccccc} 0 & \longrightarrow & A & \xrightarrow{\cdot r} & A \\ & & \big\downarrow{\scriptstyle \mathrm{id}} & \swarrow{\scriptstyle \exists} & \\ & & A & & \end{array}$$

requires the existence of $s \in A$ such that $rs = 1$. It cannot be done unless r is a unit in A. This tells us that in general A is not injective as an A-module, and puts the finger on the 'problem': for Q to be injective, we must in particular be able to extend any map $I \to Q$ from an ideal I of A to a map $A \to Q$. A nice application of Zorn's lemma shows that this property characterizes injective modules, this is precisely the Baer's criterion.

1.2. **Injecting modules into injectives.** For the rest of the time our purposes it to prove that any module injects into an injective module. This is going to be hard to prove.

1.2.1. *$\mathbb{Z}$-modules.* We start by finding at least two injective module when the ring is $\mathbb{Z}$.

Proposition 1.2. $\mathbb{Q}$ is an injective $\mathbb{Z}$-module.

Proof. By Baer's criterion we must check that for any ideal $(p) \subset \mathbb{Z}$, we can fill the following diagram:

$$\begin{array}{ccccc} 0 & \longrightarrow & p & \xrightarrow{i} & \mathbb{Z} \\ & & \big\downarrow{\scriptstyle \phi} & \swarrow{\scriptstyle \exists} & \\ & & \mathbb{Q} & & \end{array}$$

But this is very easy, it is enough to define:

$$\psi(n) := \frac{\phi(pn)}{p}.$$

□

And yet there is an other very interesting example.

Remark 4. $\mathbb{Q}/\mathbb{Z}$ is an injective $\mathbb{Z}$-module.

Proof. It is the quotient of a divisible ring. □

Now let's solve the problem over $\mathbb{Z}$, we will use a trick to solve the general problem.

Remark 5. Observe that for $\mathbb{Z}$ module M and each element $m \in M$ there is a map $M \stackrel{i_m}{\to} \mathbb{Q}/\mathbb{Z}$ such that $i_m(m) \neq 0$. This is quite easy to prove, we have a natural map $< m > \to \mathbb{Q}/\mathbb{Z}$, mapping m wherever we want if m has no annihilator, if there is n such that $n \cdot m = 0$, then choose the element $\frac{1}{n}$. Since $\mathbb{Q}/\mathbb{Z}$ is injective, we can extend this map to a map i_m.

Proposition 1.3. Any $\mathbb{Z}$ module embeds in an injective one.

Proof. Consider the map

$$M \stackrel{\prod i_m}{\longrightarrow} \prod_{m \in M} \mathbb{Q}/\mathbb{Z}.$$

Recall, product of injectives is injective. □

1.2.2. *The general case.* First of all, we find an an injective.

Remark 6. Recall that given a ring A, the set $\hom_{\mathbb{Z}}(A, \mathbb{Q}/\mathbb{Z})$ has an A-module structure with the trivial structure

$$(a \cdot f)(m) := f(am).$$

Proposition 1.4. The functor $\hom_{\mathbb{Z}}(A, \mathbb{Q}/\mathbb{Z})$ injective.

Proof. Suppose you have are in the following situation:

$$\begin{array}{ccccc} 0 & \longrightarrow & P & \xrightarrow{\;i\;} & R \\ & & \downarrow{\scriptstyle \phi} & \swarrow{\scriptstyle \exists} & \\ & & \hom_{\mathbb{Z}}(A, \mathbb{Q}/\mathbb{Z}) & & \end{array}$$

Since

$$\hom(P, \hom_{\mathbb{Z}}(A, \mathbb{Q}/\mathbb{Z})) \cong \hom(P \otimes A, \mathbb{Q}/\mathbb{Z}),$$

ψ individuates a map $\psi : P \to \mathbb{Q}/\mathbb{Z}$. Observe, now that we can solve the following problem

$$\begin{array}{ccccc} 0 & \longrightarrow & P & \xrightarrow{\;i\;} & R \\ & & \downarrow{\scriptstyle \psi} & \swarrow{\scriptstyle \rho} & \\ & & \mathbb{Q}/\mathbb{Z} & & \end{array}$$

And ρ corresponds to the desired map along the isomorphism we used before. □

Fact 1. For A module M and each element $m \in M$ there is a map $M \stackrel{i_m}{\to} \hom_{\mathbb{Z}}(A, \mathbb{Q}/\mathbb{Z})$ such that $i_m(m) \neq 0$. To prove this, just follow the same lines of the proposition above.

Proposition 1.5. Any module M embeds in an injective one.

Proof. Consider the map

$$M \stackrel{\prod i_m}{\longrightarrow} \prod_{m \in M} \hom_{\mathbb{Z}}(A, \mathbb{Q}/\mathbb{Z}).$$

Recall, product of injectives is injective. □

2. Exercises

Pay attention, exercises labelled by the tea cup ☕ may not be incredibly challenging, even not challenging, but it is important keep them in mind, so take your time when solving them and be careful to find a formal and correct solution. Exercises labelled by the danger international sign ⚠ are very challenging.

Exercise 1. if A is a field, then A itself is injective.

Exercise 2. In the following line, something is missing, can you complete it and check that it is correct?

$$\hom(P, \hom_{\mathbb{Z}}(A, \mathbb{Q}/\mathbb{Z})) \cong \hom(P \otimes A, \mathbb{Q}/\mathbb{Z}).$$