RINGS AND MODULES

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This note is going to summarize the content of the 9^{th} lesson of tutoring on the course Rings and modules. Also, attached in the end, there is an exercise sheet.

Contents

1. Class: Miscellaneous

Hello everybody. Today is a rainy day here in Brno, one of those days in which you prepare an hot chocolate and look at clouds from your warm house. We have seen and learned many things together, enough theory for us. The daily focus will be some problem solving. The theme of the exercises that I have selected come from all the lessons of the course.

Exercise 1. Let M be an A-module and $f : M \to A$ be a surjective modules. Then $M \cong \ker(f) \oplus A$.

Proof. We just need that the sequence

$$0 \to \ker(f) \to M \to A \to 0$$

splits. Recall that A is projective on itself. □

Exercise 2. $\mathbb{Q}$ is not a free $\mathbb{Z}$-module.

Proof. Suppose $\frac{a}{b}$ and $\frac{c}{d}$ are two members of a set of free generators and both fractions are in lowest terms. Find $e = \text{lcm}(b, d)$ and write both fractions as (something/e). Then

$$\frac{a}{b} = \frac{1}{e} + \cdots + \frac{1}{e} \text{ and } \frac{c}{d} = \frac{1}{e} + \cdots + \frac{1}{e},$$

where in general the numbers of terms in the two sums will be different.

Then $\frac{a}{b}$ and $\frac{c}{d}$ are not two independent members of a set of generators, since both are in the set generated by $\frac{1}{e}$. So $\mathbb{Q}$ must be generated by just one generator, so $\mathbb{Q} = \{0, \pm f, \pm 2f, \pm 3f, \ldots\}$. But that fails to include the average of f and $2f$, which is rational.

□

Exercise 3. $\mathbb{Q}$ is not a projective $\mathbb{Z}$-module.

Proof. This is just a joke. Recall that projective modules of $\mathbb{Z}$ are free! □

Exercise 4. Let A be a ring. Then the following are equivalent:

(1) Every module is projective;
(2) Every module is injective;

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(3) Every exact sequence splits.

Proof. □

Exercise 5. Injective $\mathbb{Z}$-modules are torsion free.

Proof. Suppose there is some torsion. Then there is a natural number n such that $M \to M$ has a non trivial kernel. But observe that this map is just obtained by the sequence

$$0 \to \mathbb{Z} \stackrel{\cdot n}{\to} \mathbb{Z},$$

tensoring by M. Since M is injective this sequence will still be exact, so there is no kernel. □

Exercise 6. Consider an exact sequence of modules

$$0 \to A \to B \to C \to 0.$$

Prove that a map $A \to D$ can be extended to B if and only if the exact sequence induced by pushout

$$0 \to D \to E \to F \to 0$$

splits.

Proof. Will appear soon. □

2. Exercises

Pay attention, exercises labelled by the tea cup ☕ may not be incredibly challenging, even not challenging, but it is important keep them in mind, so take your time when solving them and be careful to find a formal and correct solution. Exercises labelled by the danger international sign ⚠ are very challenging.