TOPICS IN CATEGORY THEORY

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This note is going to summarize the content of the first class of the course Topics in Category Theory.

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1. Introduction

The first two lesson of this course will be about locally finitely presentable categories. The theory of LPC offers a framework to express and relate many concepts coming from model theory and abstract algebra. It represents a fragment of the theory of accessible categories and do provide a perfect ground in which to learn many classical techniques without getting lost with set-theoretical details.

Our main reference will be the book from Rosicky and Adamek, Locally presentable and accessible categories but one can find some good introduction to the topic in the second volume of Borceux, Handbook of categorical algebra and in the old fashioned Makkai and Paré, Accessible categories: the foundations of categorical model theory.

Getting to the core of the theory will be our excuse to take a trip into some pivotal tools and concepts of category theory. We shall start by the very definition of locally finitely presentable category, which will motivate and introduce the forecoming material.

Definition 1. A category $\mathscr{K}$ is locally finitely presentable if:

- It is cocomplete;
- There is a set of finitely presentable objects $\text{Pres}_{\text{fin}}\mathscr{K}$ such that any object is a directed colimit of objects in $\text{Pres}_{\text{fin}}\mathscr{K}$.

There is no point in giving this definition so early except for making a list of things that are lacking in order to understand it. It looks like we have to introduce three important notions:

- what is a directed colimits?
- what is a finitely presentable object?

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- We should speak about very small subsets of a category having the property of shaping our category. In this case this role is played by $\mathrm{Pres}_{\mathrm{fin}}\mathcal{K}$.

To be very concise and vague, the notion of finite presentability is a measure of smallness in a category, telling us that there are some very small objects, kind of finite, *generating* the category.

2. Generators

We start our tour from the last point of the list, the concept of generator.

Definition 2. A set of objects $\mathcal{G}$ in a category $\mathcal{K}$ is a generator if it can distinguish maps, i.e. given two different parallel arrows $M \rightrightarrows N$ there exists a map $G \to M$ with $G \in \mathcal{G}$ such that the two compositions $P \to M \rightrightarrows N$ are still different.

One should not be scared by such a notion, we are very used to it. The idea behind a generator is that, for example in the category of sets, a function $X \to Y$ is completely determined by its value on elements $x \in X$.

- $\{\bullet\}$ is a generator in the category of sets;
- $\{\bullet\}$ is a generator in the category of topological spaces;
- $\mathbb{Z}$ is a generator in the category of abelian groups.

Theorem 2.1. $\mathcal{G}$ is a generator if and only if the functor

$$\coprod_{G \in \mathcal{G}} \mathcal{K}(G, \_) : \mathcal{K} \to \mathrm{Set}$$

is faithful.

Proof. A functor F is faithful precisely when $f \neq g$ implies $F(f) \neq F(g)$. So, consider two different arrows $f, g : M \to N$, since $\mathcal{G}$ is a generator there is a map $G \xrightarrow{p} M$ that distinguishes them. Now $\hom(G, f)(p) = f \circ p$, and $\hom(G, g)(p) = g \circ p$. Thus they differ on an element, namely p. □

Remark 3. It is probably better to restate this theorem in a different way, absolutely equivalent.

$\mathcal{G}$ is a generator if and only if the functor:

$$\mathcal{K} \xrightarrow{y} \mathrm{Set}^{\mathcal{G}^{\mathrm{op}}},$$

sending each object K to the functor $G \mapsto \mathcal{K}(G, K)$ is faithful.

Remark 4. A generator is a way to have a representation of a category as a subcategory of a category of presheaves or more simply as a category of sets where maps are identified with set-function. These categories are the one we are used to, Groups (for example) are just sents and homomorphism are set-function preserving the algebraic structure. This is not something that can always be done, the most remarkable example is the homotopy category of topological spaces Hot, which has not any faithful functor to Set.

Remark 5. Pay attention, the functor y is in general highly non full. In fact, it is not enough to define a function on the points on a topological space in order to extend it to a global continuous function! In this sense, one can think about a generator in a category as a dense subset of an Haussdorff space. This is a quite weak analogy but it is enough for the first day.

2.1. **Looking for generators.** A generator is useful because it provides a test set for understating the behaviour of a map. But how to find one? Suppose you have a category $\mathcal{K}$ and you are looking for a generator $\mathcal{G}$ inside $\mathcal{K}$, sometimes you can rely on free constructions.

Proposition 2.2. Let $U : \mathcal{K} \leftrightarrows \mathcal{C} : F$ be an adjoint pair such that the right adjoint $\mathcal{K} \stackrel{U}{\to} \mathcal{C}$ is faithful. If $\mathcal{C}$ has a generator, so does $\mathcal{K}$.

Proof. Suppose $\mathcal{C}$ has a generator and call it $\mathcal{G}$. We shall prove that $F(\mathcal{G})$ is a generator for $\mathcal{K}$. Choose two different maps $f, g : M \rightrightarrows N$ in $\mathcal{K}$. Since U is faithful the two maps $U(f), U(g) : U(M) \rightrightarrows U(N)$ are different. Since $\mathcal{C}$ has a generator, they are distinguished by a map $G \to U(M)$, which corresponds to a unique map $F(G) \to M$ because F is the left adjoint of U, this map distinguishes f from g. $\square$

This proposition looks a bit dumb, but it is a good way to find generators in many categories. We can use it to re-discover that:

- A-modules do have a generator, namely A.
- Groups do have a generator, namely $\mathbb{Z}$.

In fact, all these generators are the free algebra over the generator in Set, namely the one point set. This technique applies to many algebraic cases, where $\mathcal{C}$ is the category of sets and U is the forgetful functor.

2.2. **Stronger notions of generator.** To conclude the section, we shall introduce two stronger notions of generator. They will play a role later. They also show how weak is the analogy with dense subspaces in Hausdorff spaces.

Definition 6. A set of objects $\mathcal{G}$ in a category $\mathcal{K}$ is a strong generator if is a generator and moreover, given a proper subobject (monomorphism) $A \stackrel{m}{\hookrightarrow} M$, there is a map $G \to M$ with $G \in \mathcal{G}$ which does not factor trough m.

Proposition 2.3. A generator $\mathcal{G}$ is strong if and only if the functor

$$\coprod_{G \in \mathcal{G}} \mathcal{K}(G, \_) : \mathcal{K} \to \text{Set}$$

is faithful and conservative. Recall that a functor is conservative if it reflects isomorphisms.

Proof. Suppose y is conservative. And let $A \stackrel{m}{\hookrightarrow} M$, be a proper subobject. Since y is conservative $y(A) \stackrel{y(m)}{\to} y(M)$ is not an isomorphism. Since m is a monomorphism $y(m)$ is injective, thus it cannot be surjective. In conclusion there is a map $G \to M$ which is not in the image of $y(m)$, which is the thesis. For the other implication, read the proof as a shrimp would do. $\square$

Remark 7. As for remark 3, this is absolutely equivalent to the following statement:

$\mathcal{G}$ is a strong generator if and only if the functor:

$$\mathcal{K} \stackrel{y}{\to} \text{Set}^{\mathcal{G}^{\text{op}}},$$

sending each object K to the functor $G \mapsto \mathcal{K}(G, K)$ is faithful and conservative.

Definition 8. A set of objects $\mathcal{G}$ in a category $\mathcal{K}$ is a dense generator if y is full and faithful.

Remark 9. Obviously one has the following chain of implications:

$$\text{dense} \Rightarrow \text{strong} \Rightarrow \emptyset$$

When $\mathcal{K}$ has a dense generator, it can be identified with a full subcategory of a category of presheaves. Eveny more could be said about $\mathcal{K}$ if this functor would preserve colimits of limits.

Remark 10. Finally, for dense generators, a map which is defined on the *points* of an object extends to a global map. It is quite clear that this is a very strong condition.

And we shall conclude with two last (counter)examples, to introduce some zoology and get used to the new notion.

- $\{\bullet\}$ is a dense generator in Set.
- $\{\bullet\}$ is not a dense generator in Top as we observe in remark 3. In fact it is not strong neither.
- $\mathbb{Z}$ is a strong generator in the category of groups Grp, indeed a bijective homomorphism has an inverse which is an homomorphism!

3. Directed colimits

The second character of today's play is the notion of directed colimit. We shall give the crude definition and then do some effort to reduce it to a simpler one.

Definition 11. A (non-empty, for god sake) poset $(I, \leq)$ is directed if each pair of elements has an upper bound.

Definition 12. A diagram $(I, \leq) \to \mathcal{K}$ is directed if its domain is a directed poset (considered as a category).

Example 13. If λ is an ordinal, a functor $\lambda \to \mathcal{K}$ is a directed diagram. These colimits are usually called colimits of chains and they are more than the easiest example of directed colimit.

Example 14. In the category of sets the easiest example of chain is an increasing chain of sets, meaning:

$$S_1 \subset S_2 \subset \cdots \subset S_k \subset \cdots$$

Easiest does not mean trivial!

We are going to prove the following result.

Proposition 3.1. A category has directed colimits if and only if it has colimits of chains.

The proof will be a corollay of the following lemma.

Lemma 3.2. Let $(I, \leq)$ be an infinite directed poset of cardinality λ, then there exist a family of subposets $I_k \subset I, (k \leq \lambda)$ such that:

- $I = \bigcup_k I_k$;
- $I_k \subset I_{k'}$ when $k \leq k'$;
- $|I_k| < k$;
- $I_k = \bigcup_{k'<k} I_{k'}$.

Proof. Enumerate I as

$$I = \{i_k : k < \lambda\}.$$

For each finite subset $J \subset I$ choose an upper bound $j \in I$ and call $J^* = J \cup \{j\}$; for each infinite subset $L \subset I$ there exist a directed set $L^* \subset I$ of the same cardinality containing L. In fact, put $L^* := \bigcup_{n<\omega} L_n$, where L_0 is precisely L and L_{n+1} is obtained by L_n adding, for each pair of elements in L_n an upper bound. The following subposets I_k of I has the required properties:

- $I_0 = \emptyset$;

- $I_{k+1} = (I_k \cup \{i_k\})^*$;
- $I_k = \bigcup_{k'<k} I_{k'}$ for limit ordinals $k < \lambda$.

$\square$

Proof of Proposition 3.1. We need just to prove that if a category has colimits of chains, then there are directed colimits. Let $(I, \leq)$ be a directed poset and $D : I \to \mathcal{K}$ be a diagram. We proceed by transfinite induction on the cardinality of I, which we call λ.

- First step: If λ is a finite cardinal, then I has a largest element, there is nothing to prove.
- Induction: suppose the statement holds for all directed posets of cardinality less than λ. Then we use lemma 3.2: since $I = \bigcup I_k$, the colimit of D can be constructed as the colimit of the λ-chain of D_k, where D_k is the colimit of the diagram $I_k \hookrightarrow I \to \mathcal{K}$.

$\square$

All in all we learned that a directed colimit can be substituted by a colimit of a chain and a chain is a increasing family of structures in our category.

Example 15. A set S is the directed colimit of the directed syistem of its finite subsets.

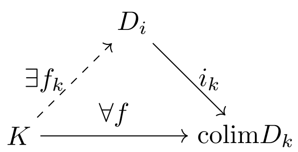
4. Finitely presentable objects

What are directed colimits useful for?

Definition 16. An object K in a category $\mathcal{K}$ is finitely presentable if the representable functor $\hom_{\mathcal{K}}(K, \_)$ preserves directed colimits, i.e.

$$\hom_{\mathcal{K}}(K, \text{colim } D) \cong \text{colim } \hom_{\mathcal{K}}(K, D(\_)).$$

Remark 17. The condition of being finitely presentable can be expressed by the diagram below.



Suppose you have a map $K \xrightarrow{f} \text{colim}D_k$, then there must be a D_i along which we can factor f. Moreover the factorization is essentially unique.

Remark 18. Now suppose that colimD_k is just a set and $\{D_k\}$ is the family of its finite subsets, if an object is finitely presentable then any map (for example monomorphisms) must factor through a finite subset, thus K must be finite. Now you see in which sense presentability is controlling the size of an object.

Example 19. A set if finitely presentable if and only if it is finite.

Proof. Let K be a finitely presentable set. K is the directed colimit of its finite subsets. If K is finitely presentable the id : $K \to K$ must factor through one of its finite subsets, which is precisely saying that K is finite. Conversely, assume that K is finite and let $K \xrightarrow{f} \text{colim}D_k$ a map into a directed colimit. Choose a point $k \in K$, its image $f(k)$ must be contained in a $D_{i(k)}$. Now just choose a D_s bigger than all $\{D_{i(k)}\}$, f must factor through D_s, because its image is in it. $\square$

Moreover one can prove that a very small colimit of very small things is still small.

Proposition 4.1. A finite colimit of finitely presentable objects is finitely presentable.

Proof. Let K_i be a finite family of finitely presentable objects and K be a colimit over this family for a shape D. We shall prove that K is finitely presentable. Consider a map $K \xrightarrow{f} \text{colim} D_k$, we can find a family of $f_i : K_i \to D_{m(i)}$

$$\begin{array}{ccc} K_i & \overset{\exists f_i}{\dashrightarrow} & D_{m(i)} \\ {\scriptstyle k_i}\downarrow & & \downarrow{\scriptstyle i_{m(i)}} \\ K & \xrightarrow{\quad f \quad} & \text{colim} D_k \end{array}$$

By directness f_i's can be chosen to have the same codomain D_m. The universal property of the colimit provides a map $K \to D_m$, which concludes the proof.

□

5. Summary

Coming back to the beginning of our class, we introduced enough notion to read the definition of locally finitely presentable category without be scared anymore.

Definition 20. A category $\mathscr{K}$ is locally finitely presentable if:

- It is cocomplete;
- There is a set of finitely presentable objects $\text{Pres}_{\text{fin}}\mathscr{K}$ such that any object is a directed colimit of objects in $\text{Pres}_{\text{fin}}\mathscr{K}$.

Remark 21. Along the class we proved that the category Set of sets is locally finitely presentable and that its finite presentables are precisely finite sets.

In the next class we will see a complete characterization and many other examples of locally finitely presentable categories.

5.1. **Spoiler.**

- The category of groups is locally finitely presentable.
- The category of $\mathbb{A}$ is locally finitely presentable.
- The category of Banach spaces is not!

6. Exercises

Pay attention, exercises labelled by the tea cup ☕ may not be incredibly challenging, even not challenging, but it is important keep them in mind, so take your time when solving them and be careful to find a formal and correct solution. Exercises labelled by the danger international sign ⚠ are very challenging.

Exercise 1. ☕ Is it possible to ping pong strong generators or dense generators as in Proposition 2.2?

Exercise 2. Prove that a topological space is finitely presentable if and only if it is finite and discrete.

Exercise 3. Prove that a category with a strong generator is well powered.