TOPICS IN CATEGORY THEORY

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This note is going to summarize the content of the first class of the course Topics in Category Theory.

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1. Introduction

Welcome to the second and last class on finitely locally presentable categories. The first class had two main character: generators and finitely presentable object.

Generators. Generators were small subcategories of the ambient category having the property of shaping all the maps, in most cases, also all the objects.

FPo. Being finitely presentable was a notion of smallness, introduced measuring the preservation of directed colimits.

With these two ideas in mind we gave the definition of locally finitely presentable category:

Definition 1. A category $\mathcal{K}$ is locally finitely presentable if:

- it is cocomplete;
- there is a set of finitely presentable objects $\mathrm{Pres}_{\mathrm{fin}}\mathcal{K}$ such that any object is a directed colimit of objects in $\mathrm{Pres}_{\mathrm{fin}}\mathcal{K}$.

And we proved that the category of sets is locally finitely presentable.

- $\mathcal{K} = \mathrm{Set}$;
- $\mathrm{Pres}_{\mathrm{fin}}\mathcal{K} = \mathrm{FinSet}$.

The lesson of today is devoted to show some other examples of LFP categories and to underline some of the properties of these categories. The structure of the lesson is the following. In the first part, since we want to find some new examples, we will look for a criterion that tells whether a category is LFP. In the second part we will prove some unexpected properties that these categories do have because of their presentability.

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2. Looking for some LFP categories

The first step in this direction is to find some necessary and sufficient condition for LFP categories.

Theorem 2.1. Let $\mathscr{K}$ be a locally finitely presentable category, then $\text{Pres}_{\text{fin}}\mathscr{K}$ is a dense generator made by finitely presentable objects.

One could prove that also the converse holds. Unfortunately as we have already seen, it is quite hard to find dense generators in a category. Thus it would be very hard to prove that a category is LFP by this technique. This is why the star of today is going to be:

Theorem 2.2. $\mathscr{K}$ is a locally finitely presentable category if and only if:

- it is cocomplete;
- there is a strong generator made by finitely presentable objects.

Thus, our technique is going to be the following. Consider a category $\mathscr{K}$ which is concrete over Set. We will try to recover a generator for $\mathscr{K}$. Here two problems comes:

- how to transport a strong/dense generator from Set on $\mathscr{K}$?
- once I have a beautiful generator, how do I know that it is made by small enough objects?

2.0.1. *Transport generators.* We shall start from solving the first issue.

Strong generators are much easier to find than dense generators. To prove so we shall state the two correspondent versions of the ping pong lemma. We will assume $\mathscr{K}$ to have equalizers.

Proposition 2.3. Let $U : \mathscr{K} \leftrightarrows \mathscr{C} : F$ be an adjoint pair such that the right adjoint $\mathscr{K} \xrightarrow{U} \mathscr{C}$ is full & faithful. If $\mathscr{C}$ has a dense generator, so does $\mathscr{K}$.

Proposition 2.4. Let $U : \mathscr{K} \leftrightarrows \mathscr{C} : F$ be an adjoint pair such that the right adjoint $\mathscr{K} \xrightarrow{U} \mathscr{C}$ is conservative and preserve monomorphisms. If $\mathscr{C}$ has a strong generator, so does $\mathscr{K}$.

Remark 2. Shall we comment why the second group of hypotheses is much easier to check then the first one:

- Notice that the forgetful functor is very hardly full, even in universal algebra. An homomorphism is not just a set function, it must preserve some structure.
- On the other side, the set-inverse of an homomorphism is very often an homomorphism, thus at least in universal algebra, the functor is likely to be conservative.
- To conclude, if every monomorphism is regular, which is the case of many categories coming from algebra, a right adjoint is going to preserve monomorphisms, because they are some special limits.

2.0.2. *Transport presentability.* This is how we are going to transport generators. Now we come to the second problem, how can we be sure that the obtained generator is made by small objects?

Definition 3. Given $F : \mathscr{K} \to \mathscr{C}$ two finitely presentable categories and a functor. F is finitely accessible if it preserves directed colimits.

Theorem 2.5. Let $U : \mathscr{K} \leftrightarrows \mathscr{C} : F$ be an adjoint pair such that the right adjoint $\mathscr{K} \xrightarrow{U} \mathscr{C}$ is finitely accessible. If $P \in \mathscr{C}$ is finitely presentable, so is $F(P)$.

Proof. Turn off your brain, you won't need it.

$$\mathcal{K}(F(P), \text{colim} D_i) \cong \mathcal{C}(P, U(\text{colim} D_i)) \cong \mathcal{C}(P, \text{colim}(UD_i)$$
$$\mathcal{C}(P, \text{colim}(UD_i) \cong \text{colim}\mathcal{C}(P, UD_i) \cong \text{colim}\mathcal{K}(F(P), D_i)$$

□

All in all, thogeter with the promised characterization in Theorem 2.7 we have the following result:

Proposition 2.6. Let $U : \mathcal{K} \leftrightarrows \mathcal{C} : F$ be an adjoint pair such that the right adjoint $\mathcal{K} \overset{U}{\to} \mathcal{C}$ is finitely accessible, conservative and preserve monomorphisms. Let $\mathcal{K}$ be cocomplete, then if $\mathcal{C}$ is finitely presentable, so is $\mathcal{K}$.

2.0.3. *The Characterization.* So now we only have to prove the characterization!

Theorem 2.7. $\mathcal{K}$ is a locally finitely presentable category if and only if:

- it is cocomplete;
- there is a strong generator made by finitely presentable objects.

Proof. There are two things to prove. The first one is that if $\mathcal{K}$ is finitely locally presentable, its set of finitely presentables is a strong generator. The second one is that, given a strong generator made by finitely presentable object $\mathcal{G}$, we shall find a set of finitely presentable objects $\mathcal{P}$ such that any object is a directed colimit of objects in $\mathcal{P}$.
Let's focus on the first one. Let M be an object and $M \rightrightarrows N$ a pair of different morphisms. Since M is a directed colimit of finitely presentables we have a family of maps $P_i \rightrightarrows M$ but the two maps coming out of M are precisely the colimit of these restrictions, so if all the restrictions coincide, they coincide too. This proves that finitely presentables are a generator. To prove that it is strong, consider a proper mono $M \hookrightarrow N$. Since N is a directed colimit of presentables $P_i \to N$, if all those maps factor through M, there is a map $N \to M$ which is going to be the inverse of the monomorphism, which is absurd.
Now part two. We will argue that

$$\mathcal{P} = \text{the closure under } \aleph_0\text{-small colimits of } \mathcal{G}$$

has the required property. For each object K we can form the canonical diagram D with respect to $\mathcal{P}$, which is the forgetful functor $D : \mathcal{P} \downarrow K \to \mathcal{K}$. Since $\mathcal{P}$ is closed under finite colimits D is filtered. Call $K^* = \text{colim} D$ and for each morphism $f : P \to K$ denote $f^* : A \to K^*$ the correspondent morphism. To conclude denote $m : K^* \to K$ the unique morphism such that $f = m \circ f^*$. We shall prove that m is an monomorphism, since $\mathcal{G}$ is a strong generator this means that it is an isomorphism. Given $q, p : B \to K^*$ with $mp = mq$ we will prove that $p = q$. It is sufficient to show the assert when $B \in \mathcal{G}$. Since D is filtered and B is finitely presentable there is a map $f : A \to K$ with $A \in \mathcal{P}$ such that both p and q factor trough f^*. That is, we have $p', q' : B \to A$ with $p = f^*p'$ and $q = f^*q'$. Let $c : A \to C$ be the coequalizer of p' and q'. Since A and B lie in $\mathcal{P}$, C lie in $\mathcal{P}$ too. Thus there is a unique $g : C \to K$ with $f = gc$. Since $f^* = (gc)^*$ we have $p = g^*cp' = g^*cq' = q$. □

2.0.4. *Examples, finally.* With this tools in our hand one can prove the following:

- Grp is locally finitely presentable.
- A-mod is locally finitely presentable.
- Abelian groups is locally finitely presentable.
- Top cannot be locally finitely presentable, its finitely presentables are finite spaces with the discrete topology.

3. Properties

In this section we want to prove what this framework is good at. We shall prove that any locally presentable category is in fact cocomplete. We will need two lemmas.

Proposition 3.1. Let $\mathscr{K}$ be a complete category and $\mathscr{G}$ a dense generator, then the the functor

$$y : \mathscr{K} \to \mathrm{Set}^{\mathscr{G}^{\mathrm{op}}}$$

has a left adjoint.

Proof. Given an object $F : \mathscr{G}^{\mathrm{op}} \to \mathrm{Set}$,let V be the category of pairs (A, a) where A is an object of $\mathscr{G}$ and $a \in FA$,with morphisms $f : (A, a) \to (A', a')$ those $\mathscr{K}$-morphisms $f : A \to A'$ which fulfil $a = Ff(a')$.The diagram given by the forgetful functor D has a colimit K. This works. □

3.1. **Limits.**

Theorem 3.2. Locally finitely presentable categories are complete.

Proof. Just use the proposition above. □