

Lesson del 27: Injectivity.

We will start and mostly concentrate on orthogonality. In some sense the main, and hidden character of this lesson is going to be the small object argument.

Recall that the category $\text{Set}^{A^{\text{op}}}$ is locally finitely presentable, in fact it has a strong generator made by f. presentable objects.

Def (orthogonality). K is orthogonal to a map $m: A \rightarrow A'$ if \forall

$$\begin{array}{ccc} A & \xrightarrow{K} & A' \\ \uparrow f & & \downarrow g \\ A & \xrightarrow{m} & A' \end{array}$$

Given a class of maps \mathcal{K} one can define \mathcal{K}^\perp .

Example

Every full reflective subcategory

$$A \begin{array}{c} \xleftarrow{L} \\ \xrightarrow{\cdot} \end{array} K$$

is an orthogonality class.

When $A = \Pi^\perp$ and Π is the set of reflections $K \rightarrow LK$

Clearly $A \subset \Pi^\perp$. To prove

the converse, let $K \in \Pi^\perp$.

$$\begin{array}{ccc} & \text{id} \rightarrow K & \\ & \swarrow & \searrow \text{ref} \\ K & \xrightarrow{m_K} & LK \end{array}$$

So m_K is a split mono and ~~also~~ and thus must be an iso.

[Ex if all reflections are mono, then they are epi].

Example

What is orthogonality useful for?

In the category Set^A one can use it to specify that a function f preserves some limit.

product: $\text{hom}(A_1 \times A_2, -) \xrightarrow{m} \text{hom}(A_1, -) \times \text{hom}(A_2, -)$

$\downarrow \quad \downarrow \quad \downarrow$
 $\Delta \quad \searrow \quad \text{F} \quad \leftarrow \quad \exists!$

Use Isoda lemma, a set transformation

$$\text{if } \text{hom}(A_1 \times A_2, -) \rightarrow \mathbb{F}$$

is an element of $\mathbb{F}(A_1 \times A_2)$!

$\mathcal{M}^\perp =$ full subcategory of $\text{Set}^\mathbb{F}$
preserving $A_1 \times A_2 \xrightarrow{\pi_i} A_i$.

Lemma

Orthogonality classes are closed under limits.

We prove it in the case of finite products:

Suppose A and $B \in \mathcal{M}^\perp$.
and consider $A \times B$

$$\begin{array}{ccc} A \times B & \xrightarrow{\pi_i} & A \text{ or } B \\ \uparrow & & \vdots \\ & & \exists! \\ \downarrow & \longrightarrow & Q \end{array}$$

and then paste.

Now we try to solve a problem, consider

$$m^L \subset \text{Set}^{A^{\text{op}}}$$

We have a very natural question, can it be reflective? We proved that all reflective ones are orthogonal after all. This question is very interesting itself, but it might look not well motivated to you. If so

① The explicit construction is done by something called small object argument, which is very useful!

② Every finitely locally presentable ~~category~~ category is a reflective subcategory of $\text{Set}^{A^{\text{op}}}$ precisely because it can be seen as an orthogonal class. Thus this is a representation theorem for \mathcal{P} -categories.

Now forget about justifications and focus on the theorem.

thm Every small orthogonality class \perp whose objects are finitely presentable is reflective of \mathcal{A} .

Idea If $\mathcal{K} = \{A \xrightarrow{m} A'\}$. We want to build, for each X a reflection

$$X \rightarrow L(X)$$

$$\begin{array}{ccc} A & \xrightarrow{m} & A' \\ f \downarrow & & \\ X & & \end{array}$$

- If m does not factorize, we take the push out

$$\begin{array}{ccc} A & \xrightarrow{m} & A' \\ \downarrow & & \downarrow \\ X & \xrightarrow{\quad} & X_0 \end{array}$$

τ_0 is the first step of our reflection.

- If it factors non uniquely we take

$$\begin{array}{ccc} A & \rightarrow & A'' \\ \downarrow & \swarrow & \\ X & & \end{array}$$

$\tau: X \rightarrow X_0$
the coequalizer

Now we iterate this construction
 if A and A' are finitely presentable
 we obtain the reflection after
 ω -steps.

Proof $\forall X$ we build a chain

$X_{ij} : X_i \rightarrow X_j$ by transfinite
 induction:

I. $X_0 = X$.

II. Isolated step: Given X_i

from a diagram

$$\begin{array}{ccc} & A & \longrightarrow A' \\ & \downarrow & \\ A' & \xrightarrow{p} & X_i \\ & \uparrow q & \end{array}$$

indexed by all the spans

$$\begin{array}{ccc} A & \longrightarrow & A' \\ \downarrow & & \\ X_i & & \end{array}$$

and all the pairs

$$A' \begin{array}{c} \xrightarrow{p} \\ \xrightarrow{q} \end{array} X_i$$

as in the idea

call X_{i+1} the colimit of
 this diagram and X_{i+1} the

$$\text{map } X_i \rightarrow X_{i+1}$$

III last step take the colimit
 of the chain.

Claim The construction stops after i_0 steps if and only if X_{i_0} is orthogonal to Π .

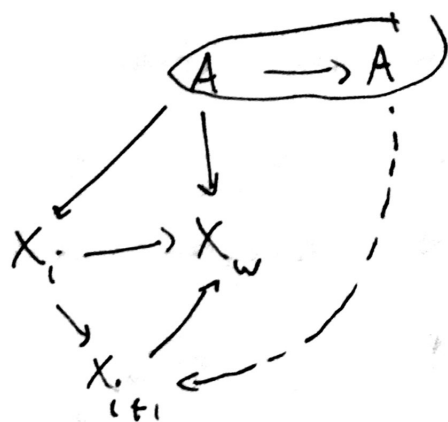
\Rightarrow If X_{i_0} is orthogonal to Π , then X_{i_0} is orthogonal. This is trivial.

\Leftarrow If it is orthogonal then it stops. This is trivial too, in fact spans \perp find in X_{i_0} a compatible color.

Claim When it stops, this in the reflection omitted.

Claim it really stops after w - steps.

We prove that X_w is in Π^\perp .



\leftarrow finitely presentable

\square .

~~Now we come to injectivity.~~

Rem In this proof it was very important that these objects in \mathcal{A} were "small" otherwise the size of X_G would increase too much.

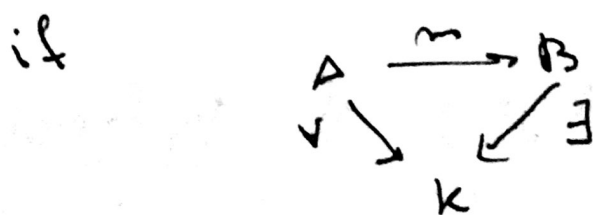
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Rem There is a strong connection between module-object argument and factorization system which

I will not talk about today.

Now we come to injectivity

Def k is injective w.r.t. $A \xrightarrow{m} B$



Rem We drop the request of uniqueness.

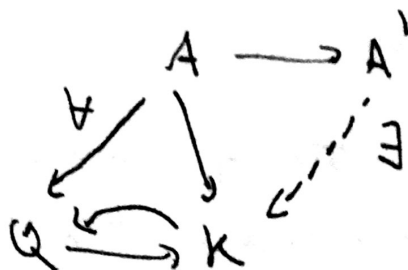
We can define injectivity class or well.

Prop

Injectivity class are closed
under product and split subobject

Proof

We only show \times split subobjects



Example

In Abelian groups $\mathcal{A} = \text{mod-}R$
one gets divisible groups.

Injectivity is very studied in
the case of Modules.

Example

In the category of topological
spaces ^{call in} the embedding of $\{0, 1\}$ in
 $[0, 1]$.

\mathcal{A} -Inj are precisely path
connected spaces.

Remark

In the context of
accessible categories is very
natural to talk about
Injectivity classes and by
small object argument one can
get as results that characterize
them as in the case
of orthogonality classes.