Lezione del 27: Injectivity.

We will start and mostly concentrate on orthogonality. In some sense the main, and hidden character of this lesson is going to be the small object argument.

Recall that the category $Set^{A^{op}}$ is locally finitely presentable, in fact it has a strong generator made by f. presentable objects.

Def (orthogonality). K is orthogonal to a map $m: A \to A'$ if $\forall$

$$\begin{tikzcd} & K & \\ A \arrow[ur, "\forall f"] \arrow[rr, "m"'] & & A' \arrow[ul, "\exists !"'] \end{tikzcd}$$

Given a class of maps $\mathcal{R}$ one can define $\mathcal{R}^{\perp}$.

Example Every full reflective subcategory

$$\begin{tikzcd} A \arrow[r, shift right, hook] & K \arrow[l, shift right, "L"'] \end{tikzcd}$$

Is an orthogonality class.

when $A = M^{\perp}$ and M is the set of reflections $K \to LK$.

Clearly $A \subset M^{\perp}$. To prove the converse, let $K \in M^{\perp}$.

$$\begin{tikzcd} & K & \\ K \arrow[ur, "id"] \arrow[rr, "m_K"'] & & LK \arrow[ul, dashed, "r"'] \end{tikzcd}$$

So m_K is a split mono and ~~also~~ and thus must be an iso.

[Ex if all reflections are mono, then they are epi].

Example What is orthogonality useful for?

In the category Set^{A} one can use it to specify that a functor F preserves some limit.

product:
$$\begin{tikzcd} hom(A_1 \times A_2, -) \arrow[r, "m"] \arrow[dr, "\forall"'] & hom(A_1, -) \times hom(A_2, -) \arrow[d, dashed, "\exists !"] \\ & F \end{tikzcd}$$

Use Yoneda lemma, a nat transformation

$$\varphi : \hom(A_1 \times A_2, -) \longrightarrow F$$

is an elmt of $F(A_1 \times A_2)$!

$\mathcal{M}^\perp$ = full subcategory of $\mathrm{Set}^{\mathcal{A}}$ preserving $A_1 \times A_2 \xrightarrow{\pi_i} A_i$.

Rem Orthogonality classes are closed under limits.

We prove it in the case of finite products:

Suppose A and $B \in \mathcal{M}^\perp$.

and consider $A \times B$

$$\begin{tikzcd}
A \times B \arrow[r, "\pi_i"] & A \text{ or } B \\
G \arrow[u] \arrow[r] & G' \arrow[u, dotted, "?\, \exists !"']
\end{tikzcd}$$

and then paste.

Now we try to solve a problem, consider

$$m^{\perp} \subset Set^{\Delta^{op}}$$

We have a very natural question, can it be reflective? We proved that all reflective ones are orthogonal after all. This question is very interesting itself, but it might look not well motivated to you. If so

① the explicit construction is done by something called small object argument, which is very useful!

② Every finitely locally presentable ~~[illegible]~~ category is a reflective subcategory of $Set^{\Delta^{op}}$ precisely because it can be seen as an orthogonality class. Thus this is a representation theorem. for P. categories.

Now forget about justifications and focus on the theorem.

<u>thm</u> Every small orthogonality class $\mathcal{M}^\perp$ whose objects of $\mathcal{M}$ are finitely presentable is reflective

<u>Idea</u> If $\mathcal{M} = \{A \xrightarrow{m} A'\}$. We want to build, for each X a reflection $X \longrightarrow L(X)$

$$\begin{tikzcd} A \arrow[r, "m"] \arrow[d, "f"'] & A' \\ X & \end{tikzcd}$$

• If m does not factorize, we take the push out

$$\begin{tikzcd} A \arrow[r, "m"] \arrow[d] & A' \arrow[d, dashed] \\ X \arrow[r, dashed, "r_0"'] & X_1 \end{tikzcd}$$

r_0 is the first step of our reflection.

• If it factors non uniquely

$$\begin{tikzcd} A \arrow[r] \arrow[d] & A'' \arrow[dl, shift left] \arrow[dl, shift right] \\ X & \end{tikzcd}$$

we take $r: X \to X_0$ the coequalizer

Now we iterate this construction

if A and A' are finitely presentable

we obtain the reflection after

ω - steps.

Proof $\forall$ X we build a chain

$x_{ij} : X_i \to X_j$ by transfinite
induction:

I. $X_0 = X$.

II. Isolated step: Given X_i

form a diagram

$$\begin{array}{ccc} & A & \longrightarrow A' \\ & f \downarrow & \\ A' & \overset{g}{\underset{q}{\rightrightarrows}} & X_i \end{array}$$

Indexed by all the spans $\begin{array}{c} A \to A' \\ \downarrow \\ X_i \end{array}$

and all the pairs $A' \overset{g}{\underset{q}{\rightrightarrows}} X_i$

as in the idea

call X_{i+1} the colimit of

this diagram and $x_{i,i+1}$ the

map $X_i \to X_{i+1}$

III limit step take the colimit
of the chain.

Claim the construction stops after i_0 steps if and only if X_{i_0} is orthogonal to Π.

$\Rightarrow$ if x_{i_0, i_0+1} is an iso, then X_{i_0} is orthogonal. this is trivial.

$\Leftarrow$ if it is orthogonal then it stops. this is trivial too, in fact spans do find in X_{i_0} a compatible cofiller.

Claim When it stops, this is the reflection. Omitted.

Claim it really stops after ω-steps.

We prove that X_ω is in $\Pi^\perp$.

$$
\begin{tikzcd}
 & A \arrow[r] \arrow[dl] \arrow[d] \arrow[r, bend left] \arrow[ddr, dashed, bend left] & A \\
X_i \arrow[r] \arrow[dr] & X_\omega & \\
 & X_{i+1} \arrow[u] &
\end{tikzcd}
$$

finitely presentable

$\square$.

~~Now we come to injectivity~~.

Rem In this proof it was very important that those objects in M were "small" otherwise the size of X_ω would increase too much.

Rem There is a strong connection between small-object argument and factorization system which I will not talk about today.

Now we can come to injectivity

Def K is injective w/r.t $A \xrightarrow{m} B$

if
$$\begin{tikzcd} A \arrow[r, "m"] \arrow[dr, "\forall"'] & B \arrow[d, "\exists"] \\ & K \end{tikzcd}$$

Rem We drop the request of uniqueness.
We can define injectivity class as well.

Prop Injectivity classes are closed under product and split subobject

Proof We only show on split subobjects

$$\begin{tikzcd}
 & A \arrow[r] \arrow[dl, "\forall"'] \arrow[d] & A' \arrow[dl, dashed, "\exists"] \\
Q \arrow[r] & K \arrow[l, bend right]
\end{tikzcd}$$

Example In Abelian groups $\mathcal{M}$ = monos one gets divisible groups.
Injectivity is very studied in the case of Modules.

Example In the category of topological spaces call $\mathcal{M}$ the embedding of $\{0,1\}$ in $[0,1]$.

$\mathcal{M}$-Inj are precisely path connected spaces.

Remark In the context of accessible categories is very natural to talk about Injectivity classes and by small object argument one can get a result that characterize them as in the case of orthogonality classes.