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ABSTRACT. This note summarizes the content of a lesson of Topics in Category Theory 2019. The main topics will be locales and localic topoi, focussing on their geometric aspects and some of their logical aspects.

1. MEET ME *there*

The aim of the following couple of lessons is to play with a fascinating family of awesomenesses. In this class, we shall enrich category theory of some geometric content, providing a suitable framework to treat geometric problems in a categorical fashion. Already in this class we will see some shades of logic appearing here and there. Next time, we will see how the *internal logic* of sets interacts with *geometry* via this framework.

Today, we sail for topos theory from the safe harbour of topological spaces. Being quite grown up as students, you should be quite convinced by now that topology encodes some relevant traits of geometry. On the other hand we should never be fooled by the easy identification of geometry with topology. The theory of topological space is a very expressive language where our intuitive notion of *light thereness* accommodates nicely, but there are many others: metric spaces, varieties, proximity spaces, *locales, topoi*, ... that might serve as a foundation to play the game of geometry.

1.1. **Topological spaces and locales.**

The first part of the lesson focuses on an alternative notion of *geometric object*, namely **locales**. Obviously, one can relate locales and topological spaces and in fact this relation restrict to an equivalence of quite relevant subcategories. Yet, a locale is a very rich object, being a poset it encodes itself a toy-category-theory, being sufficiently (co)complete there is enough structure to internalize some constructions coming from the realm of logic. As a result of this richness locales manage to relate quite different areas of mathematics, providing surprising analogies and phaenomena. The following remarks will motivate the definition of locale.

Remark 1. Let $\mathcal{X} = (X, \chi)$ be a topological space, where we indicated with X the underling set of points. The *topology* χ is a subset of $\mathcal{P}(X)$ that inherits from the boolean algebra of parts a **partial order** ($<$), **finite intersections** ($\wedge$), and **arbitrary unions** ($\bigvee$). This is in fact the very definition of topology. Moreover this poset is **infinitary distributive**, i.e.

$$U \wedge (\bigvee U_i) = \bigvee (U \wedge U_i).$$

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Remark 2. A continuous map between topological spaces $f : \mathcal{X} \to \mathcal{Y}$ induces a function $f^{-1} : \upsilon \to \chi$ mapping $V \subset \mathcal{P}(Y)$ to $f^{-1}(V) \in \mathcal{P}(X)$. Recall that f^{-1} commutes with (arbitrary) unions and intersection and thus preserves $\wedge$ and $\bigvee$.

Definition 3. A **frame** $\mathcal{O} = (O, <, \wedge, \bigvee)$ is a poset with finite limits and arbitrary colimits (when seen as a category)[1], where the infinitary distributivity law is verified. A **frame morphism** is a functor preserving finite limits and arbitrary joins.

Remark 4. As a result of the previous discussion in Rem 1 and 2, one gets a functor

$$\mathcal{O} : \mathsf{Top}^{\circ} \to \mathsf{Frm}$$

that assigns to every topological space $\mathcal{X}$ its topology $\mathcal{O}(\mathcal{X})$ and to every continuous map $f : \mathcal{X} \to \mathcal{Y}$ the map $f^* := f^{-1} : \mathcal{O}(\mathcal{Y}) \to \mathcal{O}(\mathcal{X})$ as indicated in the previous remark.

Definition 5. The category of **locales** Loc is defined as Frm°. In particular $\mathcal{O}$ can be seen as a functor $\mathcal{O} : \mathsf{Top} \to \mathsf{Loc}$.

1.1.1. *Les germes du mal.* Before going on in the study of the interaction between locales and topological spaces we shall see some properties inherited from the poset structure and inspect their geometric meaning. This is precisely where the implicit toy-category-theory starts to build an hidden connection between logic and geometry.

Proposition 1.1 (Adjoint functor theorem)**.** Let $f^* : \mathcal{O} \to \mathcal{Q}$ be a morphism of frames, then it has a right adjoint.

Proof. We shall find $f_* : \mathcal{Q} \to \mathcal{O}$ such that

$$f^*o <_{\mathcal{Q}} q \text{ iff } o <_{\mathcal{O}} f_*q.$$

Let's take this equation quite seriously and define f_* so that the previous requirement is trivially verified. Define

$$f_*(q) := \sup_{f^*o <_{\mathcal{Q}} q} o.$$

By definition, we are quite sure that if $f^*o <_{\mathcal{Q}} q$, then $o <_{\mathcal{O}} f_*q$. We need to show that if

$$o <_{\mathcal{O}} \sup_{f^*p <_{\mathcal{Q}} q} p, \text{ then } f^*o <_{\mathcal{Q}} q.$$

Since f^* preserves sups, we can apply f^* to both the sides of the inequality and get

$$f^*o <_{\mathcal{O}} f^*(\sup_{f^*p <_{\mathcal{Q}} q} p) = \sup_{f^*p <_{\mathcal{Q}} q} f^*p = q.$$

□

Remark 6. Be careful, f_* is not a morphism of frames! In the case of a continuous function $f : \mathcal{X} \to \mathcal{Y}$, the right adjoint of $f^* : \mathcal{O}(\mathcal{Y}) \to \mathcal{O}(\mathcal{X})$ corresponds to the map sending an open set U to $(\mathsf{cl} f(U^c))^c$.

Remark 7. We can use the AFT in order to prove that every frame is an Heyting algebra. A **Heyting algebra** is a lattice where given any two elements a, b there exists a greatest element x such that $a \wedge x \leq b$. This element is indicated as $a \Rightarrow b$ and is quite used by mathematicians when the Heyting algebra is the one of formulas

[1] Observe that this implies the existence of a terminal $\top$ and an initial element $\bot$.

of a certain theory. From a category theoretic point of view, this corresponds to the fact that the lattice is *cartesian closed*.

Proposition 1.2 ($\Rightarrow$)**.** Let $\mathcal{O}$ be a frame, then it is cartesian closed as a category.

Proof.

$$\_ \wedge q : \mathcal{O} \to \mathcal{O}$$

is a morphism of frames because of the infinitary distributivity law and thus has a right adjoint. This means precisely that $\mathcal{O}$ is cartesian closed. □

Remark 8 ($\neg$)**.** Consider the element $a \Rightarrow \bot$, that is the $\sup_{a \wedge x \leq \bot} x$. This is the best approximation that we can have for a complement of a in the lattice, for this reason it is denoted by $\neg a$. Pay attention, none is saying that $\neg a \vee a = \top$, this would mean that our Heyting algebra is boolean.

Exercise 1. The assignment $a \mapsto \neg\neg a$ defines a monad.

1.1.2. *Formal points.* Now we come back to our ~~topological~~ geometric interpretation of locales. We are quite happy with the intuition that a locale should be thought as the family of neighbourhoods, but since we lost our dear points, we may ask for **neighbourhoods of what precisely**?!

A part of the solution is to imagine some formal points, the same way a real number is the ideal limit of a sequence of rational numbers. The other part is to deal with this loss and overcome it. The second part of this solution is the reason for which locale theory is often called **pointless topology**.

Remark 9. A point of a topological space $\mathcal{X}$ can be thought as a (continuous) map $p : \cdot \to \mathcal{X}$. Passing to the associated locales we get a morphism of frames:

$$p^* : \mathcal{O}(\mathcal{X}) \to \mathcal{O}(\cdot).$$

Since Loc is precisely the opposite of Frm, this correspond to a morphism of frames $p : \mathcal{O}(\cdot) \to \mathcal{O}(\mathcal{X})$. Without loss of generality, p can be identified with the right adjoint p_*.

Remark 10. $\mathcal{O}(\cdot)$ coincides with the boolean algebra $2 = \{\bot < \top\}$.

Definition 11. Let $\mathcal{O}$ be a locale, the set of points of $\mathcal{O}$ is defined to be

$$\mathsf{pt}(\mathcal{O}) := \mathsf{Loc}(2, \mathcal{O})^2.$$

This set carries a natural topology. For every $o \in \mathcal{O}$ we define $\mathsf{pt}(o) \subset \mathsf{pt}(\mathcal{O})$ to be the set of those points p^* such that $p^*(u) = \top$. The family of $\{\mathsf{pt}(o)\}_{o \in \mathcal{O}}$ is a topology.

Remark 12. As a result of the previous discussion, the representable functor $\mathsf{Loc}(2, \_) : \mathsf{Loc} \to \mathbf{Set}$ lifts to a functor $\mathsf{pt} : \mathsf{Loc} \to \mathsf{Top}$.

Remark 13. It is quite natural to wonder how effective is this notion of formal point, for example, given a topological space, do we recover the underlying set if we extract the formal points of its topology? In general the answer to this question is no, the topology might make the distinction between points too blurry. Yet, pt is the conceptual inverse of $\mathcal{O}$, being its right adjoint. We shall prove that the couple of functors

$$\mathcal{O} : \mathsf{Top} \leftrightarrows \mathsf{Loc} : \mathsf{pt}$$

is an adjoint couple.

[2] That is $\mathsf{Frm}(\mathcal{O}, 2)$.

Remark 14. In order to prove the adjunction we need to observe that $\mathcal{O}$ is representable too. That's not hard to see, observe that $\mathcal{O}(\mathcal{X}) \cong \mathsf{Top}(\mathcal{X}, \mathsf{S})$, where we indicated with S the Sierpiński space. The correspondence that associates to a map $f : \mathcal{X} \to \mathsf{S}$ the open set $f^{-1}(\top)$[3] extends to an isomorphism of locales.

Remark 15. Being representable, $\mathcal{O}$ will preserve colimits[4].

Proposition 1.3. $\mathcal{O} \dashv \mathsf{pt}$ is an adjoint couple:

Proof. We shall concentrate on the following line of abstract nonsense.

$$\mathsf{Loc}(\mathcal{O}(1), \mathcal{Q}) =: \mathsf{pt}(\mathcal{Q}) \cong \mathsf{Top}(1, \mathsf{pt}(\mathcal{Q})).$$

This means that $\mathcal{O}$ has the universal property of the left adjoint on the terminal object. Since 1 is colimit-dense[5] in Top, for every topological space $\mathcal{X}$ there is a diagram $D_{\mathcal{X}}$ such that $\mathcal{X} \cong \mathrm{colim}_{D_{\mathcal{X}}} 1$. We conclude using that $\mathcal{O}$ preserves colimits.

$$\begin{aligned}
\mathsf{Top}(\mathcal{X}, \mathsf{pt}(\mathcal{Q})) &= \mathsf{Top}(\mathrm{colim}_{D_{\mathcal{X}}} 1, \mathsf{pt}(\mathcal{Q})) \\
&= \lim_{D_{\mathcal{X}}} \mathsf{Top}(1, \mathsf{pt}(\mathcal{Q})) \\
&= \lim_{D_{\mathcal{X}}} \mathsf{Loc}(\mathcal{O}(1), \mathcal{Q}) \\
&= \mathsf{Loc}(\mathrm{colim}_{D_{\mathcal{X}}} \mathcal{O}(1), \mathcal{Q}) \\
&= \mathsf{Loc}(\mathcal{O}(\mathrm{colim}_{D_{\mathcal{X}}} 1), \mathcal{Q}) \\
&= \mathsf{Loc}(\mathcal{O}(\mathcal{X}), \mathcal{Q}).
\end{aligned}$$

□

Remark 16. We do not have enough time to describe in detail what kind of topological space can be recovered from its topology, those are called sober space. I just hope to have convinced you that a locale is a valid alternative to topology, where we have some geometric intuition and we can can recover our naive pointset topology by studying formal points. Surprisingly, if we keep our geometric intuition on locales, even without mentioning points, we manage to prove a lot of classical results, this proves that somehow *thereness* is not related to points, we can formally express *nearness* even if we do not precisely know what's near what.

1.2. **Locales and localic topoi.** In this section we introduce localic topoi. A localic topos is going to be a very strange object, even quite unmotivated for today. In this lesson, the best that we will manage to prove is that localic topoi are an equivalent framework to locales, thus a localic topos is as good as a locale to do some geometry.

Definition 17. Let $\mathcal{O}$ be a locale. A sheaf on a locale is a functor $S : \mathcal{O}^\circ \to \mathbf{Set}$ with the additional property that if $\{o_i\} \subset \mathcal{O}$ is a family closed under finite intersection, then $S(\mathrm{colim}\ o_i) = \lim S(o_i)$.

Remark 18. The most known example of sheaf over a locale is the **sheaf of continuous functions**. Let $\mathcal{M}$ be a manifold. We define $C[\_, \mathbb{R}] : \mathcal{O}(\mathcal{M})^\circ \to \mathbf{Set}$ mapping an open set to the sets of its real valued continuous function. The sheaf condition, also called *descent*, corresponds to the fact that the local data of a compatible family of continuous functions extends to a globally defined continuous function.

[3] $\top$ is the only open point of S.

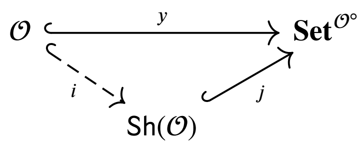
[4] Be careful with the (co)variance!

[5] Be careful, it is not dense!

In the case of a quite large family of topological spaces this sheaf encodes most of the relevant topological properties of the space. For this reason people started to consider sheaves as a tool to study topology. Eventually Grothendieck *decided* that these objects were geometric in first place.

Definition 19. We define the **category of sheaves** $\mathsf{Sh}(\mathcal{O})$ over a locale $\mathcal{O}$ to be the full subcategory of sheaves in $\mathbf{Set}^{\mathcal{O}^\circ}$. A **localic topos** is a category that is equivalent to a category of sheaves over a locale.

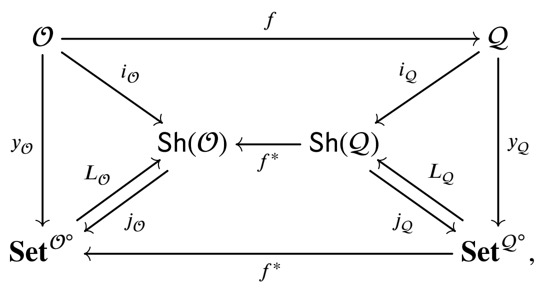
Remark 20. Recall that that Yoneda embedding, $y_{\mathcal{O}} : \mathcal{O} \to \mathbf{Set}^{\mathcal{O}^\circ}$ is a fully faithful functor. Moreover, every object in the image of y is a sheaf because representable functors preserve all limits, thus y factorizes along $\mathsf{Sh}(\mathcal{O})$.



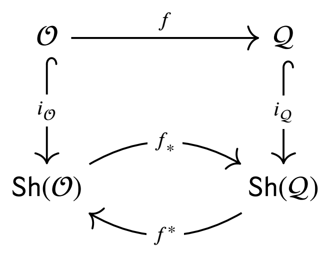
Proposition 1.4. $\mathsf{Sh}(\mathcal{O})$ is reflective in $\mathbf{Set}^{\mathcal{O}^\circ}$ and the left adjoint $L_{\mathcal{O}}$ preserves finite limits.

Proof. Omitted. We really do not have the time to prove this statement. □

Remark 21. In order to motivate the notion of morphism of localic topoi, let $f : \mathcal{Q} \to \mathcal{O}$ be a morphism of locales. Recall that this is precisely that data of a morphism of frames in the opposite direction. The map f^* (that exists at the level of presheaf categories) maps sheaves to sheaves and is cocontinuous. And thus can be lifted to the categories of sheaves.



Moreover, the previous discussion states precisely that f^* coincides with the composition $L_{\mathcal{O}} \circ f^* \circ j_{\mathcal{Q}}$ and thus preserve finite limits. Indeed $f^* \circ j_{\mathcal{Q}}$ preserve all limits[6], while $L_{\mathcal{O}}$ preserves finite limits.



[6] f^* is both continuous and cocontinuous.

Finally, by the AFT[7] f^* has a right adjoint f_*. Thus, for every *continuous* function we can generate an adjunction $f^* \dashv f_*$ where the left adjoint preserves finite limits.

Definition 22. A **geometric morphism** of localic topoi $f : \mathcal{G} \to \mathcal{E}$ is the data of an adjunction $f^* : \mathcal{E} \leftrightarrows \mathcal{G} : f_*$ whose left adjoint preserve finite limits. Observe that the morphism has the direction of the right adjoint.

Remark 23. As a result of the previous discussion we get a functor

$$\mathsf{Sh} : \mathsf{Loc} \to \mathsf{LocTopoi}$$

assigning to a locale $\mathcal{O}$ its category of sheaves $\mathsf{Sh}(\mathcal{O})$, and to a morphism of locales f the adjoint couple $f^* \dashv f_*$.

Remark 24. We managed to give a kind of geometric interpretation to some localic topoi and some of the maps that relate them. The quest is still very far from being over, we promised that the two framework are equivalent...

Remark 25 (Sh is essentially surjective)**.** A question now comes quite natural, let $\mathcal{E}$ be a localic topos, is it possible to recover the locale from which it comes from? To answer this question we need an intrinsic description of the locale in terms of the category-theoretic internal language topos. We have already seen that $\mathcal{O}$ is hidden somewhere in $\mathsf{Sh}(\mathcal{O})$, via a factorization of the Yoneda embedding. How do we find it?!

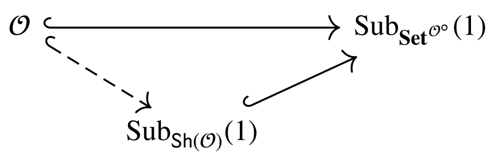
Observe that $\mathcal{O}$ coincides with the lattice of subobjects of its terminal object, i.e.

$$\mathcal{O} \cong \mathrm{Sub}_{\mathcal{O}}(\top).$$

That's in fact quite trivial to observe. Now, since the Yoneda embedding preserves limits, $y(\top)$ coincides with the terminal object of $\mathbf{Set}^{\mathcal{O}^\circ}$, and since y preserve monomorphisms[8], we get an injective function

$$\mathcal{O} \cong \mathrm{Sub}_{\mathcal{O}}(\top) \to \mathrm{Sub}_{\mathbf{Set}^{\mathcal{O}^\circ}}(1).$$

Since $\mathcal{O}$ sits fully faithfully in $\mathsf{Sh}(\mathcal{O})$, this inclusion factors through $\mathrm{Sub}_{\mathsf{Sh}(\mathcal{O})}(1)$.



We shall prove that the dotted arrow is a bijection. Let $S \to 1$ be a subsheaf of 1. Given an open set $o \in \mathcal{O}$, obviously we have $S(o) = 1$ or $S(o) = \emptyset$. Thus we can define

$$u = \sup_{o : S(o) = 1} o.$$

Since S is a sheaf, $S(u) = 1$ and it is quite evident that $y(u)$ coincides with S on every open set, thus S was representable in first place, as desired.

Corollary 1.5. $\mathcal{O} \cong \mathrm{Sub}_{\mathsf{Sh}(\mathcal{O})}(1)$.

[7]Since f^* is cocontinuous, to apply the AFT it's enough to provide a generator of $\mathsf{Sh}(\mathcal{O})$. On the other hand it is quite easy to see that a fully reflective subcategory of a category with a dense generator, has a dense generator.

[8]It preserves all limits.

Remark 26 (Sh is full)**.** Now we know how to knock on the door of a localic topos and ask for the locale it comes from. The language spoken by the topos is quite categorical. We still feel quite far from proving that localic topoi are the same of locales, geometric morphisms might be much more that those coming from a map of locales. In fact that's not true. Given a geometric morphism of localic topoi

$$f^* : \mathcal{E} \leftrightarrows \mathcal{G} : f_*,$$

the map f^*, preserves monomorphisms and terminal objects[9]. In particular its restriction to $\mathrm{Sub}_{\mathcal{E}}(1)$ lands in $\mathrm{Sub}_{\mathcal{G}}(1)$. The fact that it is cocontinuous and preserve finite limits ensures that the restriction is a morphism of frames and thus f^* corresponds to a morphism of locales $f : \mathrm{Sub}_{\mathcal{G}}(1) \to \mathrm{Sub}_{\mathcal{E}}(1)$.

Remark 27 (The functor $\mathrm{Sub}_{\_}(\mathbf{1})$)**.** We can pack together the previous discussion in the single observation that

$$\mathrm{Sub}_{\_}(1) : \mathsf{LocTopoi} \to \mathsf{Locales}$$

is a functor. Moreover in the previous remarks we essentially proved that the couple $(\mathrm{Sub}_{\_}(1), \mathsf{Sh})$ yields an equivalence of categories.

Theorem 1.6.

$$\mathrm{Sub}_{\_}(1) : \mathsf{LocTopoi} \to \mathsf{Locales} : \mathsf{Sh}$$

is an equivalence of categories.

Remark 28. The class is over, we did quite a bit of mathematics today. We introduced the notion of locale and the notion of localic topos and we saw how they relate with the notion of topological space. Now we know that localic topoi can serve as generalized spaces and we can keep a geometric intuition on them as long as we study geometric morphisms. In the next lesson we will unveil a deep logical content of localic topoi, somehow guided by the same kind of observations of the remarks 1.2 and 8.

[9]It preserves finite limits.