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ABSTRACT. This note summarizes the content of a lesson of Topics in Category Theory 2019. The main topics will be locales and localic topoi, focussing on their logical aspects.

CONTENTS

1. *There* IS A WAY TO *be*

In the previous lesson we studied localic topoi. A localic topos can be understood as a generalized space and corresponds to the category of sheaves over the locale of its subobjects of 1,

$$\mathcal{E} \cong \mathsf{Sh}(\mathsf{Sub}_{\mathcal{E}}(1)).$$

In this lesson we will concentrate on the foundational properties of a (localic) topos. The general motto of today should be that a *localic topos is a universe of sets*. In order to understand the meaning of this sentence, we should try to agree of what a universe of sets should be. The first part of the lesson will be devoted to understand some properties that characterize the category of sets.

1.1. Set: a case study.

Remark 1. Without listing precisely the axioms of ZFC, it is not hard to recognize the importance of some properties of the category of sets.

(1) Given two sets a, b, one can form the product $a \times b$.
(2) Given two sets a, b one can form the set of functions $\mathbf{Set}(a, b)$.
(3) $a \cong \mathbf{Set}(1, a)$ for all sets.
(4) Given a set a one can form the powerset $\mathcal{P}(a)$, cointaining all the subsets of a.
(5) There is an internal understaning of the poweset, in the sense that $\mathcal{P}(a) \cong \mathbf{Set}(a, 2)$. Observe that this is an *internal* understating because $\mathbf{Set}(a, 2)$ is a set.

Remark 2. The fact that $\mathbf{Set}(a, b)$ is a set itself has a deep conceptual meaning. We are labelling with a special type an object that has no reason, in principle, to belong to **Set**. For example in the category of groups, the hom-sets ares not a groups. This

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axiom is internalizing to the category of sets an object that would belong to some meta-theoretic world. The same is true for products and powerset.

Remark 3. Two other properties that are quite relevant in set theory are:

(1) The terminal object 1 is a strong generator.
(2) There is an infinite set.

We are not sure that these properties have a conceptual meaning,The forst one looks more related to make easier computations and arguments. The second one is used to make our set theory more expressive.

1.1.1. **Set** *is finitely complete and cartesian closed.*

Remark 4. As a result of the previous discussion, we shall observe that (1), (2), (3) are just instances of the fact that **Set** is a cartesian closed category with finite limits. Recall that a category $\mathcal{E}$ is cartesian closed when for every $\boldsymbol{B}$, the functor $\_ \times b : \mathcal{E} \to \mathcal{E}$ has a right adjiont $\_^b$, i.e.

$$\mathcal{E}(a \times b, c) \cong \mathcal{E}(a, c^b).$$

In particular we obtain that $a \cong a^1$ via Yoneda lemma:

$$\mathcal{E}(\_, a^1) \cong \mathcal{E}(\_ \times 1, a) \cong \mathcal{E}(\_, a).$$

And this correspond to the fact that a set is the same of its *internal points*.

1.1.2. **Set** *has a subobject classifier.*

Remark 5. It is well known that there is a bijection

$$\chi : \mathcal{P}(a) \leftrightarrows 2^a : \_^{-1}(1),$$

where the map χ is sending a subset $m : b \subset a$ to the function $\chi_m : a \to 2$ that is 1 over the elements of b and 0 elsewhere. The inverse $\_^{-1}(1)$ maps f to $f^{-1}(1)$, as suggested by the chosen notation. Observe that $f^{-1}(1)$ coincides with the pullback of the following diagram.

$$\begin{array}{ccc} & & 1 \\ & & \downarrow{\scriptstyle t} \\ a & \xrightarrow[f]{} & 2 \end{array}$$

Remark 6. Observe that if we think about 2 as the powerset of 1, the map t is pointing at the maximal subobject of 1 in its poset of subobjects.

Definition 7. Let $\mathcal{E}$ be a category. A **subobject classifier** is an object Ω together with a monomorphism $t : 1 \to \Omega$ such that such that every subobject $m : b \to a$ in $\mathcal{E}$ is the pullback of this morphism along a unique morphism[1] $\chi_m : a \to \Omega$.

$$\begin{array}{ccc} b & \longrightarrow & 1 \\ {\scriptstyle m}\downarrow & & \downarrow{\scriptstyle t} \\ a & \xrightarrow[\chi_m]{} & \Omega \end{array}$$

[1]The characteristic morphism of m.

Remark 8. $\mathcal{E}$ has a subobject classifier if and only if the functor Sub : $\mathcal{E}^\circ \to$ **Set** is representable and in that case we have

$$\text{Sub} \cong \mathcal{E}(\_, \Omega),$$

as in the case of **Set**. To be more precise, Sub : $\mathcal{E}^\circ \to$ **Set** is the functor that assigns to an object a its class of subobjects. For the action of Sub on maps, let $f : a \to b$ be a morphism and $m : c \to b$ be a subobject. Then $\text{Sub}(f)(m)$ is defined via the following pullback.

$$\begin{array}{ccc} p & \longrightarrow & c \\ {\scriptstyle \text{Sub}(f)(m)}\downarrow & & \downarrow{\scriptstyle m} \\ a & \xrightarrow[f]{} & b \end{array}$$

Remark 9. Observe that the last rephrasing of having a subobject classifier corresponds to an *exactness property* of **Set**. Informmally, an exactness property of a category is a compatibility law between limits and colimits. The infinitary distributivity law in a locale is an exactness condition of the underlying poset. The fact that the functor Sub is representable implies that is maps colimits into limits. In a concrete case, this means that if $a = \text{colim}\, a_i$ and $b \to a$ is a subobject of a, then b is the colimit of the pullbacks $a_i \cap a$.

$$\begin{array}{ccc} \text{colim}\, b \cap a_i & \longrightarrow & b \\ \downarrow & & \downarrow \\ \text{colim}\, a_i & \longrightarrow & a \end{array}$$

This is a compatibility between colimits and pullback along monomorphisms. Keep in mind that in the case of locales, this exactness condition is precisely where the geometry is hidden in the abstract algebra of the definition.

1.1.3. *Elementary topoi.* With the previous section in mind we are ready to give the precise definition of *a category of sets*.

Definition 10. An elementary topos $\mathcal{E}$ is a finitely complete and cartesian closed category with a subobject classifier.

Of course, this does not mean that any elementary topos is equivalent to the category of sets, it just means that we provided enough structure to play with the basic construction of sets.

1.2. **Localic topoi are elementary.** The rest of the class will be devoted to prove that a localic topos is elementary. We will need a technical result that already appeared in the previous lesson. One maight think that our aim is to prove that a localic topos is as good as **Set** to run mathematics, instead our aim is to convince the reader that there is some internal logic going on in a localic topos. Be careful, because there is something deep in this somewhat technical statement.

Theorem 1.1. The category $\mathsf{Sh}(\mathcal{O})$ is reflective in $\mathbf{Set}^{\mathcal{O}^\circ}$ and the reflector $L_{\mathcal{O}}$ preserves finite limits[2].

[2]Some people say the reflector is *left exact*.

Remark 11. Again, quite informally, one can transport along the reflector any exactness condition of the presheaf category that involves finite limits and arbitrary colimits (because this is the structure preserved by such a reflector). This is exactly what happens in the case of locales, where the infinitary distributivity law is conceptually inherited from the fact that it inhabits a totally distributive boolean algebra, namely a powerset. Does it mean that the geometry of the localic topos is encoded in the fact that the reflection is left exact? *Yes, it does.*

1.2.1. $\mathsf{Sh}(\mathcal{O})$ *is finitely complete.*

Lemma 1.2. $\mathsf{Sh}(\mathcal{O})$ is finitely complete.

Proof. This is the easiest thing to prove. $\mathsf{Sh}(\mathcal{O})$ sits inside $\mathbf{Set}^{\mathcal{O}^\circ}$ and is closed under limits in this presheasf category. In fact this proves that $\mathsf{Sh}(\mathcal{O})$ is complete. □

1.2.2. $\mathsf{Sh}(\mathcal{O})$ *is cartesian closed.*

Remark 12. Let P, Q be two sheaves, we need to define P^Q. Since we have no clue about the possible definition, it might be a good idea to start from a strong generator. For every $o \in \mathcal{O}$, we want to give a definition such that

$$\mathsf{Sh}(\mathcal{O})(y(o) \times Q, P) \cong \mathsf{Sh}(\mathcal{O})(y(o), P^Q),$$

But recall that, whatever P^Q is defined to be, the Yoneda lemma impies that

$$\mathsf{Sh}(\mathcal{O})(y(o), P^Q) = P^Q(o).$$

Thus we can use the last one as a definition and hope for the best,

$$P^Q(\_) = \mathsf{Sh}(\mathcal{O})(y(\_) \times Q, P).$$

Now, we should show that P^Q is a sheaf with this definition and that this sheaf has the desired property. Instead of following this path we will provide a formal argument for which the category is cartesian closed, by uniqueness of the right adjoint the functor that we will prove to exist has to match with the one that we just defined.

Lemma 1.3. $\mathsf{Sh}(\mathcal{O})$ is is cartesian closed.

Proof. This proof should remind you about how we prove that a frame is an Heyting algebra. We will show that the functor $\_ \times a : \mathsf{Sh}(\mathcal{O}) \to \mathsf{Sh}(\mathcal{O})$ preserve colimits, and thus is a left adjoint via the AFT. In order to accomplish this task, let's give some names. Recall that $\mathsf{Sh}(\mathcal{O})$ is reflective in $\mathbf{Set}^{\mathcal{O}^\circ}$,

$$L_{\mathcal{O}} : \mathbf{Set}^{\mathcal{O}^\circ} \leftrightarrows \mathsf{Sh}(\mathcal{O}) : j_{\mathcal{O}}.$$

Now, the functor $\_ \times j_{\mathcal{O}}(a) : \mathbf{Set}^{\mathcal{O}^\circ} \to \mathbf{Set}^{\mathcal{O}^\circ}$ preserve all colimits, because $\mathbf{Set}^{\mathcal{O}^\circ}$ inherits cartesian closedness from $\mathbf{Set}$. Let D be a diagram in $\mathsf{Sh}(\mathcal{O})$

$$\begin{aligned} (\text{colim} D) \times a &\cong (L_{\mathcal{O}} \text{colim} j_{\mathcal{O}} D) \times a \\ L_{\mathcal{O}} \text{ preserves finite limits } &\cong L_{\mathcal{O}}((\text{colim} j_{\mathcal{O}} D) \times j_{\mathcal{O}} a) \\ \mathbf{Set}^{\mathcal{O}^\circ} \text{ is cartesian closed } &\cong L_{\mathcal{O}}(\text{colim}(j_{\mathcal{O}} D \times j_{\mathcal{O}} a)) \\ L_{\mathcal{O}} \text{ preserves colimits } &\cong \text{colim}(L_{\mathcal{O}} j_{\mathcal{O}} D \times a) \\ &\cong \text{colim}(D \times a). \end{aligned}$$

Observe that it was enough to use that $L_{\mathcal{O}}$ preserve finite products in the proof.

□

1.2.3. $\mathsf{Sh}(\mathcal{O})$ *has a subobject classifier.* The strategy is always the same, first we prove that $\mathbf{Set}^{\mathcal{O}^\circ}$ has a subobject classifier, then we show that having a subobject classifier is stable under left exact reflections.

Remark 13. We discussed in advance that a category $\mathcal{E}$ has a subobject classifier when the functor $\mathrm{Sub} : \mathcal{E}^\circ \to \mathbf{Set}$ is representable, i.e.

$$\mathrm{Sub}(\_) \cong \mathcal{E}(\_, \Omega).$$

We should try to see if this observation helps us to guess the definition of this functor. With the Yoneda lemma in mind, we evaluate the previous expression on a representable,

$$\mathrm{Sub}(y(o)) \cong \mathcal{E}(y(o), \Omega) = \Omega(o).$$

And we obtained the only possible definition of the functor Ω. Now we should prove that it has the desired universal property.

Remark 14. We should also specify a universal true map $t : 1 \to \Omega$. This is the map picking, for each o the maximal subobject in $\mathrm{Sub}(y(o))$. Observe that is was precisely what we where doing in the case of $\mathbf{Set}$, as pointed out in Rem 6.

Remark 15. Recall that $\mathbf{Set}$ itself is the localic topos over the terminal topological space. In this case the functor Ω is identified with its image (because a functor $1 \to \mathbf{Set}$ is an object of $\mathbf{Set}$) and coincides with $\mathrm{Sub}(1) = \mathcal{P}(1) = 2$, as expected.

Remark 16. Before proving that the omega that we just defined is a subobject classifier, observe that we can give a better description of some of its elements, in fact:

$$\{p \in \mathcal{O} : p \leq q\} \cong \mathrm{Sub}_{\mathcal{O}}(o) \subset \mathrm{Sub}_{\mathsf{Sh}(\mathcal{O})}(y(o))$$

Lemma 1.4. $\mathsf{Sh}(\mathcal{O})$ has a subobject classifier.

Proof. Let $m : Q \to P$ be a monomorphism in $\mathsf{Sh}(\mathcal{O})$, we need to prove that there exists a unique natural transformation χ^m that forms a pullback like this one:

$$\begin{array}{ccc} Q & \longrightarrow & 1 \\ {\scriptstyle m}\downarrow & & \downarrow{\scriptstyle t} \\ P & \xrightarrow[\chi^m]{} & \Omega \end{array}$$

For every object o of $\mathcal{O}$ and every element $x \in P(o)$, we know that this χ^m has to be a natural transformation such that $\chi^m_o(x)$ is a certain subfunctor of $y(o)$,

$$\begin{array}{ccc} Q(o) & \longrightarrow & 1 \\ {\scriptstyle m}\downarrow & & \downarrow{\scriptstyle t} \\ P(o) & \xrightarrow[\chi^m_o]{} & \{p : p \leq o\} \end{array}$$

As a result of this discussion, for every $x \in P(o)$, $\chi^m_o(x)$ should be a family of opens smaller that o that checks if x is of the form $m(y)$ for some y in $Q(o)$. Now define,

$$\chi^m_o(x) = \{g \leq o : P(g) \subset Q(o)\}.$$

No doubts that this is a subobject of $y(o)$, in fact it is a very special selection of maps into o. One can check that this choice works. □

1.3. Grothendieck topoi.

Remark 17. When we study topological spaces we know that from an algebraic point of view, the geometry of the object is hidden in an exactness condition of the poset of open sets, namely the infinitary distributivity law. The geometry of the maps stays encoded in the preservation of intersections and joins.

Remark 18. Even if we did not prove it, one can believe that this compatibility of finite intersection with arbitrary union propagates to the category of sheaves over a locale. This coincides precisely with the fact that the reflector $L_{\mathcal{O}} : \mathbf{Set}^{\mathcal{O}^\circ} \to \mathsf{Sh}(\mathcal{O})$ preserves finite limits. Thus the geometry of a localic topos is completely encoded in this compatibility of finite limits and arbirary colimits. One can say the same for geometric morphism, where the geometric behaviour is was encoded in the preservation of ~~finite intersections and arbitrary joins~~ finite limits and arbitrary colimits.

Definition 19. A geothendieck topos $\mathcal{G}$ a a left exact localization of a presheaf category. A morphism of Grothenddieck topoi $f : \mathcal{E} \to \mathcal{G}$ is a functor $f^* : \mathcal{G} \to \mathcal{E}$ preserving colimits and finite limits.

Theorem 1.5. A Grothendieck topos is an elementary topos.

Proof. Omitted, but highly hinted in the previous proofs. □

Remark 20. As a final remark observe that a a presheaf category is an elementary topos essentially because **Set** is so, instead a Grothendieck topos is elementary because its geometric axioms (the reflector being left exact) impose enough compatibility between limits and colimits to preserve the elementary logic of the presheaf category. This means that the defining axioms of **Set** that we care about are inherently geometric. If we require more internal logic of the category of sets to be preserved, the reflector will make some obstructions, or equivalently, *the geometry will not care*.