TOPOLOGY

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ABSTRACT. This note summarizes the content of the first lesson of tutoring on the course Topology 2019. Also, attached at the end, there is an exercise sheet.

1. WHAT IS A GEOMETRIC OBJECT?

Definition 1. A *topological space* is a couple $\mathcal{X} = (X, \chi)$ where X is a set and $\chi \subset \mathcal{P}(X)$ is a family of parts stable under arbitrary union and finite intersection containing the all set and the empty set.

(1) χ is the *topology*.
(2) an element $O \in \chi$ is an *open* set.
(3) the complement of an open is a *closed* set.
(4) Since the complement of a set individuates the set itself, a topology could be given by its family of closed sets. Of course this family is stable under arbitrary intersection and finite union.
(5) In this note the topology of the space will be always indicated with the corresponding greek letter. Thus, for example, χ will be the topology of $\mathcal{X}$ and υ will be the topology of $\mathcal{Y}$.

Definition 2. A *continuous function* $f : \mathcal{X} \to \mathcal{Y}$ is a function such that respects the topology, i.e. $f^{-1}(\upsilon) \subset \chi$.

The aim of our first meeting is to understand something of these obscure definitions. Topology was introduced with the aim of clarifying real analysis and eventually led to our standard concept of geometric object. A geometric object should be an hack, possibly a set, equipped with a suitable notion of *thereness*. Probably the most successful definition of geometric object in the direction of compromising expressiveness with intuitiveness is the one of *metric space*, that we recall.

Definition 3. A *metric space* is a couple $\mathcal{M} = (M, d)$ where M is a set and $d : M^2 \to \mathbb{R}$ is a function with the following properties.

M1 $d(x, x) = 0$
M2 $d(x, y) > 0$ (if $x \neq y$).
M3 $d(y, x) + d(x, z) \geq d(y, z)$.

Remark 4. Ah-ah! I tricked you already! Haven't I forgotten the symmetry condition?! Be careful because that's precisely the aim of this class: understanding why a metric space is a geometric object. This means that we do not really care about all its properties.

Remark 5. In the definition of metric space we use our understaing of real numbers to establish if two points are far. Something is *here* if it is not far from me [M1], something is *there* if it's very far from me [M2], and - of course - never go from Brno to Prague passing trough Paris [M3]! I will assume you studied and understand the notion of metric space in your past and I will try to use the notion of metric space as a shelter to justify the definition of topological space.

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Remark 6. We use some properties of real numbers to navigate a metric space. For instance, since $\mathbb{R}$ is ordered we can say that something is further than something else. Since it is complete we can emulate the idea of approaching to a point. All in all, there are no doubts that the most relevant concepts in metric spaces are *adherence* and *convergence*. You might even not remember the definitions of the two, but you probably will remember that most of the proofs in the theory of metric spaces are based on a suitable variation of the sentence *the point x is very close to the set A*. That's the case of a sequence $A = \{x_n\}$ converging to a point x. With these ideas in mind we can define a relation on a metric space as follows.

Definition 7 (Nearness in metric spaces)**.** Let $\mathcal{M} = (M, d)$ be a metric space. Its nearness relation $\Subset_d \subset M \times \mathcal{P}(M)$ is the subset of couples (x, A) such that $d(x, A) = 0$. Recall that the distance $d(x, A)$ is defined to be

$$d(x, A) \coloneqq \inf_{y \in A} d(x, y).$$

If (x, A) belongs to $\Subset_d$ we say that x is near A and we write $x \Subset_d A$ in a more conventional infix notation.

Remark 8 (Nearness and continuity)**.** Continuity in metric spaces can be defined via the ϵ/δ nonsense (which is not very intuitive) or using convergence of sequences. In both cases it is quite easy to see that one can rewrite to continuity condition in terms of the nearness relation as follows. A function between metric spaces $f : \mathcal{M} \to \mathcal{L}$ is continuous if and only if whenever $x \Subset_{\mathcal{M}} A$ then $f(x) \Subset_{\mathcal{L}} f(A)$. This means precisely that f preserves nearness, isn't this one of the most intuitive formulations of continuity?

Proof. If f preserves nearness, it must be continuous. Indeed if $(x_n)_{n\in\mathbb{N}}$ converges to x, then $x \Subset \{x_n\}_{n\in\mathbb{N}}$. Since f preserves nearness, $f(x) \Subset \{f(x)_n\}_{n\in\mathbb{N}}$, that means precisely that f preserves converging sequences. For the other implication, assumes that f is continuous and $x \Subset A$. Since $d(x, A) = 0$ there must be a sequence $(x_n)_{n\in\mathbb{N}} \in A$ such that $\lim_n d(x, x_n) = 0$. Since f is continuous, $\lim_n d(f(x), f(x_n)) = 0$ and thus $f(x) \Subset f(A)$. □

Remark 9 (Nearness and closure operators)**.** Recall that a metric space $\mathcal{M}$ comes equipped with a closure operator $\mathrm{cl} : \mathcal{P}(M) \to \mathcal{P}(M)$ defined by the formula:

$$\mathrm{cl}(A) \coloneqq \{x \in M : d(x, A) = 0\}.$$

It is essentially a tautology to say that $x \Subset A$ if and only if $x \in \mathrm{cl}(A)$. This observation sets a very important conceptual link between closures and nearness relations.

Now we come to an axiomatization of the nearness relation in a metric space. Keeping in mind its definition in the metric case we find every axiom extremely natural.

Definition 10. Let X be a set. A *nearness* relation $\Subset$ on X is a subset $\Subset \subset X \times \mathcal{P}(X)$ with the following properties.

N1 no element is near the empty set.
N2 if $x \in A$ then x is near A.
N3 if x is near $A \cup B$ then it is near A or it is near B.
N4 if x is near A and every element of A is near B, then x is near B.

To be formal, we rewrite one of the axioms of the previous list in Bourbakist style: [N3] can be reformulated as $x \Subset A \cup B \Rightarrow x \Subset A$ or $x \Subset B$.

Remark 11. A nearness relation establishes a notion of *fuzzy membership*, the structure believes that an element belongs to a set ($x \Subset A$), even if it does not actually sit inside it.

Example 12 (Metric nearness is a nearness relation)**.** Just for some sanity check, we shall see that the metric nearness is indeed a nearness relation.

N1 is completely evident.
N2 is true because of [M1], that is $d(x,x)=0$.
N3 is due to the fact that

$$\inf_{y\in A\cup B} d(x,y) = \min\{\inf_{y\in A} d(x,y), \inf_{y\in B} d(x,y)\}.$$

Indeed the left hand side is $\leq$ then the right hand side. On the other hand, if the LHS < RHS, then there is some $y \in A\cup B$ such that:

$$\inf_{x\in A\cup B} d(x,y) \leq d(x,y) < \min\{\inf_{y\in A} d(x,y), \inf_{y\in B} d(x,y)\}.$$

But then $d(x,y)$ is strictly smaller than both the infs which is clearly impossible because y has to belong either to A either to B.
N4 relies on the fact that if x is near every element of A and every element of A is near B, then $x \in \mathrm{cl}(A) \subset \mathrm{cl}(B)$ and thus $x \Subset B$.

Example 13 (A trivial example)**.** Another nearness relation that in fact is metrizable but does not have a metric spirit in first place is defined by

$$x \Subset A \text{ iff } x \in A.$$

Example 14 (Geometric objects might not be metric)**.** We find the concept of nearness much weaker than the notion of distance and probably more suitable for some concrete geometric examples. The set of functions $\mathbb{R}^{\mathbb{R}}$ can be endowed with a very natural nearness relation:

$$f \Subset A \text{ iff for all } x\in\mathbb{R}, \inf_{g\in A} |f(x)-g(x)| = 0.$$

This nearness relation is known in analysis as the topology of pointwise convergence and provides a geometric intuition to handle analytical problems. There is no metric that induces this nearness relation. This is not hard at all to prove, but I will reserve this exercise for a class on *countability axioms*.

Remark 15. In the same fashion of the metric case, a nearness relation $\Subset$ on X induces a closure operator $\mathrm{cl}_{\Subset} : \mathcal{P}(X) \to \mathcal{P}(X)$ which is defined by $\mathrm{cl}_{\Subset}(A) \coloneqq \{x \in X : x \Subset A\}$ with the following properties:
K1 $\mathrm{cl}_{\Subset}(\varnothing) = \varnothing$;
K2 if $A \subset \mathrm{cl}_{\Subset}(A)$
K3 $\mathrm{cl}_{\Subset}(A\cup B) = \mathrm{cl}(A)\cup\mathrm{cl}(B)$;
K4 $\mathrm{cl}^2_{\Subset} = \mathrm{cl}_{\Subset}$.
An operator with these properties is called *Kuratowski closure operator*, and probably you already proved in your life that the closure operator associated to a metric space has these properties. In this case, all come as a very natural consequence of the notion of nearness relation, for example [N1] implies [K1], [N2] implies [K2], [N3] implies [K3]. Finally [N4] implies [K4], even if this last one is not completely evident.

Hopefully I convinced you that nearness is a suitable way of providing a set with a geometric structure. Now we come to the main result of this class, which was motivating the notion of topology.

Theorem 1.1 (Nearness is topology)**.** There is a bijection between nearness relations on a set X and topologies on X.

Proof. We define explicitly the correspondence, mapping $\chi \mapsto \Subset_\chi$ and $\Subset \mapsto \chi_{\Subset}$. Let χ be a topology on X, we say that $x \Subset_\chi A$ if and only if $x \notin \bigcup_{O\subset A^c} O$. Now let $\Subset$ be a nearness relation, we say that $O \in \chi_{\Subset}$ if and only if $\mathrm{cl}_{\Subset}(O^c)$ is fixed by $\mathrm{cl}_{\Subset}$. The proof that these two function are one the inverse of the other is not locally trivial and globally technical. Let's prove together that one composition is the identity. We shall prove that $\chi = \chi_{\Subset_\chi}$. It is enough to prove that C is a closed set in χ if and only if $\mathrm{cl}_{\Subset_\chi}(C) = C$, in fact this is just a rephrasing of the thesis. To accomplish this task we compute explicitly $\mathrm{cl}_{\Subset_\chi}(C)$,

$$\mathrm{cl}_{\Subset_\chi}(C) = \{x \in X \,:\, x \notin \bigcup_{O \subset C^c} O\}.$$

Now we observe that when C is closed, $\bigcup_{O \subset C^c} O$ coincides with C^c and thus $\mathrm{cl}_{\Subset_\chi}(C) = \{x \notin C^c\} = C$. When $\mathrm{cl}_{\Subset_\chi}(C) = C$, C must be closed, because $\mathrm{cl}_{\Subset_\chi}(C)$ is (by definition) the complement of a union of open sets. □

Remark 16. The class is over, I hope I gave enough ideas to justify the intuition behind the notion of topological space, the nearness relation (which is a very natural concept) induced by a topology is described explicitly in the last theorem, but I recommend to draw some pictures and get used to its geometric meaning. As a side remark, I hope to have convinced you that the concept itself of geometric entity is evolving within the time and can be approached from different perspective (metric spaces, nearness relations, topologies, closure operators, proximity spaces, varieties, manifolds, even topoi or simplicial sets...), what stays solid in this constant change is our fascination for geometry itself and our need for a geometric intuition when picturing or facing a problem (think, for example, about the standard proof of Cauchy Lipschitz theorem).

References

[1] Vectornaut (https://mathoverflow.net/users/1096/vectornaut), *Why is a topology made up of open sets?*, MathOverflow, URL:https://mathoverflow.net/q/19173 (version: 2015-10-23).

[2] user58514, *What is the topology of point-wise convergence?*, Mathematics Stack Exchange, URL:https://math.stackexchange.com/q/293004 (version: 2013-02-02).

2. Exercises

Pay attention, exercises labelled by the tea cup ☕ may not be incredibly challenging, even not challenging, but it is important keep them in mind, so take your time when solving them and be careful to find a formal and correct solution. Exercises labelled by the danger international sign ⚠ are very challenging. Exercise labelled by 📖 come from the beautiful book **Elementary Topology Problem Textbook**, by Viro, Ivanov, Netsvetaev and Kharlamov.

Exercise 1. Prove that the family of closed sets is closed under arbitrary intersection and finite union.

Exercise 2 (☕). Prove that a function between topological spaces $f : \mathcal{X} \to \mathcal{Y}$ is continuous if and only if it preserves the nearness relation induced by the topology.

Exercise 3. Complete the main theorem, showing that $\in\, = \,\in_{\chi_{\in}}$.

The Book (📖). This week we focus mainly on metric spaces and their first topological properties.

4.A
4.B
4’2
4’3
4’4
4’9
4’10