TOPOLOGY

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ABSTRACT. This note summarizes the content of the 10th lesson of tutoring on the course Topology 2019. Also, attached at the end, there is an exercise sheet.

1. USING COVERINGS

Recall that the main result of the previous lesson was a connection between groups acting on a simply connected space $G \curvearrowright \mathcal{E}$ and the fundamental group $\pi_1(\mathcal{E}/G)$ of the topological quotient.

Theorem 1.1 (Main theorem)**.** Let $G \curvearrowright \mathcal{E}$ be a properly discontinuous action of a group on a (path connected) simply connected space $\mathcal{E}$. Then $G \cong \pi_1(\mathcal{E}/G, x)$.

We used this result to provide a proof that the fundamental group of the circle is $\mathbb{Z}$, that represents the building block of the whole elementary algebraic topology.

Corollary 1.2. The fundamental group of the circle is $\mathbb{Z}$.

Remark 1. The proof was based on the observation that $\mathbb{Z}$ acts on the real line by translation. Then, since the real line is contractible (and thus is simply connected), we can apply the theorem and get the the fundamental group of the quotient space (the circle) is $\mathbb{Z}$.

Remark 2. This proof-line suggests a strategy: given a topological space we can use coverings to *blow up* some homotopical singularities and encode them in a action of a group over the unsingularized space (the covering). In this lesson we use this strategy to:

(1) compute the fundamental group of a projective space;
(2) prove that every group occurs as a fundamental group of some space.

1.1. **The fundamental group of the projective space.**

Fact 1. The sphere S^n is simply connected.

Corollary 1.3. The fundamental group of a projective space $\mathbb{RP}^n$ is $\mathbb{Z}_2$, the cyclic group with two elements.

Proof. Recall that the topological data of $\mathbb{RP}^n$ can be presented as the quotient of the sphere S^n under the action of $\mathbb{Z}_2$,

$$[n] \cdot p = (-1)^n p.$$

Since the sphere is simply connected we can deduce that the fundamental group of $\mathbb{RP}^n$ is $\mathbb{Z}_2$. □

Remark 3. For the last time, we split the data of $\mathbb{RP}^n$ in two: $(\mathsf{S}^n, \mathbb{Z}_2)$. One is a homotopically simpler space, S^n, the other is a group. The homotopical information about $\mathbb{RP}^n$ is encoded in the action. This idea suggests that **a space is always the same as a simply connected cover together with an appropriate action of its fundamental group.**

Remark 4. Today we can't prove that the sphere S^n is simply connected, and in fact this will be precisely the content of the next lesson. *One should always leave the table a bit hungry.*

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1.2. **Every group if the π_1 of its Δ-realization.** This part of the class is devoted to prove that every group is the fundamental group of some space. We prove the statement under the additional assumption that the group is finite, with few adjustments the proof generalizes straightforward to the infinite case, but one should be careful about some technical details that would obscure the beauty of the argument.

Remark 5. We know the strategy, we want a simply connected space over which G acts nicely, then we apply our theorem.

Remark 6. Let G be a finite group of cardinality n. $\mathbb{R}[G]$ is the free vector space over G. This means that an element $v \in \mathbb{R}[G]$ is (uniquely represented by) a formal sum

$$v = \sum_{g_i \in G} \lambda_i \cdot g_i$$

for some real numbers λ_i.

Remark 7. G acts on $\mathbb{R}[G]$ in a very natural way (by multiplication):

$$g \cdot v = g \cdot \left(\sum_{g_i \in G} \lambda_i \cdot g_i \right) := \sum_{g_i \in G} \lambda_i \cdot gg_i.$$

$\mathbb{R}[G]$ is simply connected (and in fact is contractible) thus we feel on the right track, but observe that this action is not properly discontinuous, in fact it has a fixed point, i.e. 0.

Remark 8. On the other hand there is a very nice subspace of $\mathbb{R}[G]$ on which the action is properly discontinuous, and that is $\Delta[G]$. $\Delta[G]$ is defined to be the following subset of $\mathbb{R}[G]$.

$$\Delta[G] := \left\{ \sum_{g_i \in G} \lambda_i \cdot g_i \in \mathbb{R}[G] \, : \, \sum \lambda_i = 1 \text{ and all } \lambda_i \geq 0 \right\}.$$

Observe that this set does not contain 0 and since G is cancellative it is quite easy to show that the restricted action $G \curvearrowright \Delta[G]$ is properly discontinuous.

Fact 2. $\Delta[G]$ is contractible.

Corollary 1.4. The fundamental group of the quotient space $\Delta[G]/G$ is G.

Proof. $\Delta[G]$ is simply connected, G acts nicely and wolves howl at the moon... □

Remark 9. So, we are very powerful, we know how to compute many fundamental groups. But how the hell do we prove that S^n is simply connected? And in general, **if a space is simply connected, but is not contractible, how do we show it?** A strategy to attack this problem will be the main character of our last lesson in algebraic topology.

2. Exercises

Pay attention, exercises labelled by the tea cup may not be incredibly challenging, even not challenging, but it is important keep them in mind, so take your time when solving them and be careful to find a formal and correct solution. Exercises labelled by the danger international sign are very challenging. Exercise labelled by come from the beautiful book **Elementary Topology Problem Textbook**, by Viro, Ivanov, Netsvetaev and Kharlamov.

Remark 10. All the exercises are mandatory.

Exercise 1. Show that the restricted action $G \curvearrowright \Delta[G]$ is properly discontinuous.

Exercise 2. Prove Fact 2.