TOPOLOGY

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ABSTRACT. This note summarizes the content of the 10th lesson of tutoring on the course Topology 2019. Also, attached at the end, there is an exercise sheet.

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The last lesson left an open problem. Given a non contractible space that we feel is simply connected, how do we show it?! How favourite example of this phenomenon is the sphere, which is quite far from being contractible.

Remark 1. The lesson of today provides a very effective and completely alternative technique to coverings to compute fundamental groups and in particular show that some spaces are simple connected.

Remark 2. The general idea is quite simple let $\mathcal{X}$ be a space of which we want to compute the fundamental group and imagine that we have a nice splitting of the space in two subspaces $\mathcal{Z}$ and $\mathcal{Y}$ of which we do understand the fundamental group such that $\mathcal{X} = \mathcal{Z} \cup \mathcal{Y}$, can we recover the $\pi_1(\mathcal{X})$ from the fundamental groups of $\mathcal{Z}$ and $\mathcal{Y}$? The answer will turn out to be true.

Remark 3. Intuitively, since $\mathcal{X} = \mathcal{Z} \cup \mathcal{Y}$, we could hope for a surjective map $\mathcal{Z} \coprod \mathcal{Y} \twoheadrightarrow \mathcal{X}$, and thus we expect to see a surjective map $\pi_1(\mathcal{Z}) \coprod \pi_1(\mathcal{Y}) \twoheadrightarrow \pi_1(\mathcal{X})$, whatever the symbol $\coprod$ means among groups.

Remark 4. Thus in this lesson we have essentially two things to do,

(1) the first one is to develop a suitable notion of join of groups.
(2) the second one is to exploit it in order to get informations on the fundamental group of $\mathcal{Z} \cup \mathcal{Y}$.

1.1. Free product of groups.

Remark 5. Let G, H be two groups. Then the set $G \star H$ is the quotient of set of all finite words in the elements of G and H

$$g_{11}g_{12} \cdots g_{1n_1}h_{11} \cdots h_{1n_2} \cdots g_{n1} \cdots g_{nn_n}h_{n1} \cdots h_{nn_n}$$

under the equivalence relation that identifies two words if we can apply the multiplication of some of the two groups. For example, imagine that $g_1g_2 = g_3$ in G, then

$$g_1g_2h_1 = g_3h_1.$$

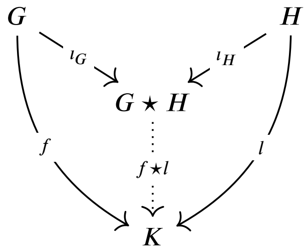
Remark 6. $G \star H$ has a very natural group structure obtained by just-apposition of words. The identity of $G \star H$ is the empty word.

Remark 7. There is an injective map from both G and H into $G \star H$, sending (say) g to the atomic word containing only g. This map is evidently a group homomorphism. We indicate these maps with the names ι_G and ι_H.

Remark 8. Observe that G is always isomorphic to $G \star 1$, where 1 is the trivial group with just one element.

Date: 6 April 2019.

Proposition 1.1. Given two group homomorphism $G, H \to K$ there is a unique extension to the free product that matches the natural inclusions.



Proof. The proof is quite simple. We define

$$(f \star l)(g_{11}g_{12} \cdots g_{1n_1}h_{11} \cdots h_{1n_2} \cdots g_{n1} \cdots g_{nn_n}h_{n1} \cdots h_{nn_n})$$

to be defined element-wise, in the sense of the following line

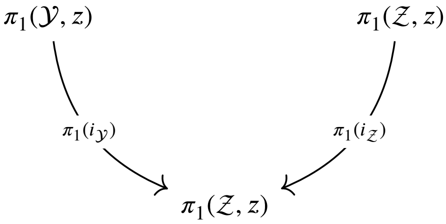
$$f(g_{11})f(g_{12}) \cdots f(g_{1n_1})l(h_{11}) \cdots l(h_{1n_2}) \cdots f(g_{n1}) \cdots f(g_{nn_n})l(h_{n1}) \cdots l(h_{nn_n}).$$

Obviously last just-apposition is a product is computed in K. Observe that this is completely necessary because we want that $(f \star l)\circ\iota_{G/H} = f/g$ and the images of ι_G and ι_H generate $G \star H$ under product. □

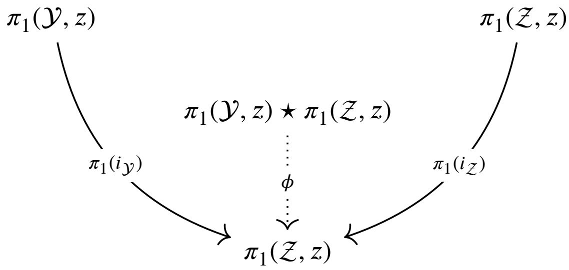
1.2. **Van Kampen theorem.**

Remark 9. Given $i : (\mathcal{Z}, z) \subset (\mathcal{X}, z)$ a (pointed) subspace, we get a map $\pi_1(i) : \pi_1(\mathcal{Z}, z) \to \pi_1(\mathcal{X}, z)$. But be careful, this map is far from being injective in general, for one example, see the inclusion of the circle in the disk.

Remark 10. Given two subspaces $(\mathcal{Y}, z)$ and $(\mathcal{Z}, z)$ contained in $(\mathcal{X}, z)$, we get two maps:



Observe that z has to lie in the intersection $\mathcal{Y} \cap \mathcal{Z}$ for this to have sense. Thus, by the description of the free product, we get a map



But please, do not get carried away, this abstract nonsense cannot generate a theorem, for example ϕ is not always surjective. For example, write $\mathcal{X}$ as the union of its points, if such a ϕ were always onto, then any space would be simply connected.

Theorem 1.2 ((Weak) Van Kampen)**.** Let $\mathcal{X} = \mathcal{Y} \cup \mathcal{Z}$ be a space written as the unions of two open subsets. Assume moreover that $\mathcal{Y}$, $\mathcal{Z}$ and $\mathcal{Y} \cap \mathcal{Z}$ are path connected. Then the map ϕ described above is onto. (Recall that the base point has to be chosen in the intersection).

Proof. Please, see **59.1** in **Topology** by **Munkres**. In class we will follow his proof line by line. □

Corollary 1.3. Spheres are simply connected.

Proof. Write S^n as $\mathsf{S}^+ \cup \mathsf{S}^-$, where S^+ is the set

$$\mathsf{S}^+ = \{(\bar{x}, z) \,:\, z > -\epsilon^1\},$$

and S^- is defined analogously. Observe that both of them are contractible and satisfy the hypotheses of (w)VK, thus $\pi_1(\mathsf{S}^n)$ admits a surjection from the trivial group with one element, and that means that it has to be trivial. □

[1] ϵ should be positive and very close to 0.

2. EXERCISES

Pay attention, exercises labelled by the tea cup may not be incredibly challenging, even not challenging, but it is important keep them in mind, so take your time when solving them and be careful to find a formal and correct solution. Exercises labelled by the danger international sign are very challenging. Exercise labelled by come from the beautiful book **Elementary Topology Problem Textbook**, by Viro, Ivanov, Netsvetaev and Kharlamov.

Remark 11. All the exercises are mandatory.

Definition 12 (The topologist bag)**.** The topologist bag B is a subspace of $\mathbb{R}^3$ and is a sphere together with the diameter connecting the north pole to the south pole.

Exercise 1 (The topologist bag is a bag)**.** Prove that the topologist bag is homeomorphic to a sphere together with a path connecting the north pole to the south pole that does not lie inside the sphere.

Exercise 2. Compute the fundamental group of the topologist bag.

Exercise 3. Exhibit a simply connected covering of the topologist bag.