TOPOLOGY

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ABSTRACT. This note summarizes the content of the second lesson of tutoring on the course Topology 2019. Also, attached at the end, there is an exercise sheet.

1. BESTIARY

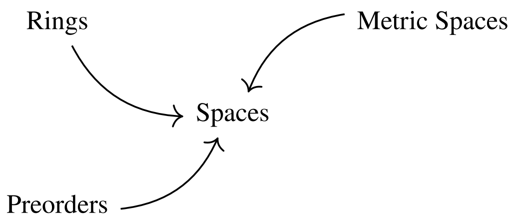
Welcome back to our course in topology. In the first lesson we focused on the definition of topological space and we tried to provide an intuition for such gadgets. Our motivation was mostly rooted in the theory of metric spaces.

Remark 1. According to the main theorem of the first lesson, a topology on X is the data of a closure operator on $\mathcal{P}(X)$ that generalizes the proprieties of the closure operator associated to a metric space.

The lesson of today has two main targets:

(1) Take a *safari* trough some examples of topological space;
(2) See how we can encode some interesting notions into geometric concepts.

To be more precise on the second aim of this lesson, we have already seen how every metric space is topological. Today we will see how to associate a topological space to any preorder and to any ring. Surprisingly (or maybe not), order-preserving maps and ring homomorphism have geometric representation.



Before going on, we recall some notations and definitions from the previous lesson and we remark some features of these definitions.

Definition 2. A *topological space* is a couple $\mathcal{X} = (X, \chi)$ where X is a set and $\chi \subset \mathcal{P}(X)$ is a family of parts stable under arbitrary union and finite intersection containing the all set and the empty set.

(1) The space will be always indicated with the corresponding greek letter. Thus, for example, χ will be the topology of $\mathcal{X}$ and υ will be the topology of $\mathcal{Y}$.
(2) χ is the *topology*.
(3) an element $O \in \chi$ is an *open* set.
(4) the complement of an open is a *closed* set.
(5) A topology could be given by its family of closed sets. Of course this family is stable under arbitrary intersection and finite union.
(6) To every topological space $\mathcal{X} = (X, \chi)$ one can associate a closure operator cl_χ : $\mathcal{P}(X) \to \mathcal{P}(X)$, mapping a set A to the smallest closed set containing A.
(7) In a topological space $\mathcal{X} = (X, \chi)$ a point x is *near* a set A if and only if $x \in \mathrm{cl}(A)$.

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CONTENTS

1.1. **Discrete & indiscrete topology.**

Definition 3. Let X be a set. The *discrete topology* on X is the topology in which every element of $\mathcal{P}(X)$ is declared to be open.

Remark 4. It is worthy to say that this topology is induced by a metric, that is

$$d(x,y) = \begin{cases} 0 \text{ if } x = y \\ 1 \text{ if } x \neq y. \end{cases}$$

In fact the closure of a set $A \subset X$, that is defined to be the set of points whose distance from A is 0 is A itself, thus every set is closed with this metric. Observe that a different metric might induce the same topology, it would be enough to say that $d(x, y) = 5$ when $x \neq y$. This means that it might happen that two metrics on the same set induce the same topology.

Remark 5. Let $\mathcal{X} \to \mathcal{Y}$ be a function. When the domain is a set endowed with the discrete topology, every function is a continuous function. This is again completely trivial, because the counterimage of any set will be open.

Definition 6. Let X be a set. The *indiscrete topology* on X is the topology in which only the empty set and the whole set are declared to be open.

Remark 7. There is no metric that induces this topology! In fact, when a topology is induced by a metric, given two points $x, y \in \mathcal{X}$ there are always two open sets O_x, O_y that do not intersect such that $x \in O_x$ and $y \in O_y$. In these cases one say that the topology distinguishes points, or that the topology is *Hausdorff*, or that the topology is T2. Obviously the indiscrete topology is not T2.

Remark 8. Let $\mathcal{X} \to \mathcal{Y}$ be a function. When the codomain is a set endowed with the discrete topology, every function is a continuous function. This is again completely trivial, because the counterimage of any open set will be open.

1.2. **The real line $\mathbb{R}$.** We will not spend much time on this example, because somehow it is the one that you already know the best. The standard topology on the real numbers is the one induced by the standard metric. To state it clearly, we declare that a set is closed if and only if it is closed with respect to the metric,

$$d(x, y) = |x - y|.$$

We will use this topology to see some useful counterexamples.

(1) A set A might be nor closed, nor open. That is the case of $[0, 1)$.
(2) A set A might be both, closed and open. That is the case of the empty set. In this topological space only the empty set and the whole set are both closed and open.
(3) The arbitrary intersection of open sets is not open, in fact

$$[0,1) = \bigcap_{n>0} \left(-\frac{1}{n}, 1\right).$$

1.3. **Cofinite topology & polynomials on $\mathbb{R}$.**

Definition 9. Let X be a set. The *cofinite* topology on X is the topology whose closed sets are finite subsets of X.

Remark 10. Indeed this is a topology, the arbitrary intersection of finite sets is still finite, and so are their finite unions.

Remark 11. On a finite set, this topology coincides with the discrete topology.

Remark 12. Let's study the cofinite topology $(\mathbb{R}, \mathsf{CF})$ on the real numbers. I claim that a polynomial function $p : (\mathbb{R}, \mathsf{CF}) \to (\mathbb{R}, \mathsf{CF})$ is always continuous with respect to this topology. Indeed the counterimage of a point is always a finite set, because

$$p^{-1}(y) = \{x : p(x) - y = 0\},$$

and $p(x) - y$ is a polynomial having at most $\deg(p)$ solutions. More generally, the counterimage of a finite set A is the union of the counterimage of its points, that is thus a finite union of finite sets, this shows that p is continuous.

1.4. **Alexandroff spaces & specialization topology.**

Definition 13. Let $\mathcal{X}$ be a topological space. We say that $\mathcal{X}$ is *Alexandroff* if open sets are stable under arbitrary intersection.

Remark 14. Alexandroff spaces are rare.

(1) The (in)discrete topology is Alexandroff;
(2) The cofinite topology is never Alexandroff on an infinite set;
(3) the standard topology on real numbers is not Alexandroff;
(4) if a topology induced by a metric is Alexandroff, then it is discrete.

Thus, it is quite natural to wonder what's the generic shape of an Alexandroff space. Fortunately we have a very natural example of Alexandroff spaces.

Definition 15. Let $(P, \leq)$ be a preordered set[1]. The specialization topology (or *Alexandroff topology*) on P is defined as follows: $A \subset P$ is open if and only if it is upwards closed, i.e. if $x \in A$ and $x \leq y$ then $y \in A$.

Remark 16. The specialization topology on a preorder is always an Alexandroff topology. Indeed the arbitrary union and intersection of upwards closed sets is upwards closed.

Remark 17. A function $f : (P, \leq) \to (Q, \leq)$ is order-preserving if and only if it is a continuous map with respect to the Alexandroff topology.

Theorem 1.1. Any Alexandroff space comes from a poset endowed with the Alexandroff topology.

Proof. Let $\mathcal{A} = (A, \tau)$ be an Alexandroff space. We equip A with a preorder. Say $a \leq b$ if and only if $a \in \mathrm{cl}(b)$. If you still remember something of nearness relations, we are saying that $a \leq b$ if and only if $a \Subset b$. We claim that $\mathcal{A}$ is precisely $(A, \leq)$ equipped with the Alexandroff topology Alex. In order to prove it, we show that the closure operators cl_τ and $\mathrm{cl}_{\mathsf{Alex}}$ are the same function. Since both topologies are Alexandroff, the closure operators preserve arbitrary unions, i.e.

$$\mathrm{cl}(\bigvee A_i) = \bigvee \mathrm{cl}(A_i).$$

In particular, since every set is the union of its elements, cl_τ and $\mathrm{cl}_{\mathsf{Alex}}$ are the same function if and only if the coincide on singletons. To finish, observe that $\leq$ is defined precisely in order to make this happen. □

[1]This means that $\leq$ is reflexive and transitive.

1.5. **The spectrum of a ring.**

Definition 18. Let A be a ring. The *spectrum* of a ring $\mathsf{Spec}(A)$ is the set of is prime ideals[2]. Let I be an ideal of A, we define $V(I) \subset \mathsf{Spec}(A)$ to be the set of all primes containing I. We endow $\mathsf{Spec}(A)$ with the topology whose closed sets are $V(I)$, for I an ideal of A. This is known as the Zariski topology Z on $\mathsf{Spec}(A)$.

Remark 19 (Sanity check)**.** Z is a topology. In fact:

(1) $V(I) \cup V(J) = V(IJ)$
(2) $\bigcap_k V(I_k) = V(\sum I_k)$

Remark 20. A ring homomorphism $A \to B$ induces a continuous function

$$f^{-1} : \mathsf{Spec}(B) \to \mathsf{Spec}(A)$$

between their spectra endowed with the Zariski topology.

[2]Recall that an ideal p is prime if, whenever a product $a \cdot b$ belongs to p one of the two elements a, b belongs to p.

2. Exercises

Pay attention, exercises labelled by the tea cup ☕ may not be incredibly challenging, even not challenging, but it is important keep them in mind, so take your time when solving them and be careful to find a formal and correct solution. Exercises labelled by the danger international sign ⚠ are very challenging. Exercise labelled by 📕 come from the beautiful book **Elementary Topology Problem Textbook**, by Viro, Ivanov, Netsvetaev and Kharlamov.

Exercise 1. Going back to Rem. 12, prove that if a topology τ on the real number makes all the polynomials $p : (\mathbb{R}, \tau) \to (\mathbb{R}, \mathsf{CF})$ continuous, then $\mathsf{CF} \subset \tau$. Observe that, together with Rem. 12 this means that CF is the smallest topology making polynomials continuous.

Exercise 2. Is the Zariski topology T2?

Exercise 3. Prove Rem. 19.

Exercise 4 (☕). Prove Rem. 20.

The Book (📕). We see some basics of the definition of topology and continuous functions.

2’2	2.B
2’4	2.4,5,6,7,8.
2’12x	Only 2.Jx’s
10’2	
10’3	