TOPOLOGY

IVAN DI LIBERTI

ABSTRACT. This note summarizes the content of the third lesson of tutoring on the course Topology 2019. Also, attached at the end, there is an exercise sheet.

1. SEPARATION AXIOMS

Welcome back to our course in topology. I hope you learnt something in these four lessons of introduction to topology, because today it's your day. We will focus separation axioms, examples, counterexamples and simple exercises. The use of the word *separation axioms* for separation properties of topological spaces is not accidental. Depending on what kind of topological spaces a mathematician typically dealt with, he had a corresponding notion of *decency* for topological spaces. Every separation property gives different *proof techniques.* The lesson of today will be mostly interactive.

1.1. Kolmogorov, seu T_0.

In lesson 2, while studying Alexandroff spaces we were very close to the definition of topological indistinguishability.

Definition 1. Two points x, y in a topological space $\mathcal{X}$ are topological indistinguishable $x \equiv y$ if x is near y and viceversa[1].

We observed that topological indistinguishability is an equivalence relation and we used it in order to construct natural examples of Alexandroff spaces. In fact, in that case we were using a variation of indistinguishability, that is called specialization preorder.

Exercise 1. A space is T_0 if and only if the only point which is topological indistinguishable from x is x itself.

Given a topological space $\mathcal{X}$ we can endow $\mathcal{X}/\equiv$ with the biggest topology $\tau_\equiv$ making the quotient map $\pi : \mathcal{X} \to \mathcal{X}/\equiv$ continuous.

Exercise 2 (Kolmogorification)**.** $\mathcal{X}/\equiv$ is universal among T_0-approximations of $\mathcal{X}$, in the sense that:

(1) $\mathcal{X}/\equiv$ is T_0.
(2) Every continuous map $\mathcal{X} \to \mathcal{Y}$, where $\mathcal{Y}$ is a T_0-space factors trough $\mathcal{X}/\equiv$.

Sketch of proof of (2). When a function is continuous, it preserves nearness. This means that two equivalent points in $\mathcal{X}$ are sent in equivalent points in $\mathcal{Y}$. Since the image T_0, f in constant on $\equiv$-equivalence classes. Thus there is a set-theoretical factorization of f along $\pi : \mathcal{X} \to \mathcal{X}/\equiv$, this function is continuous. □

Example 2. Most of the topological spaces in nature are much more than T_0. For this reason we just make a list of **non** T_0-spaces:

(1) the indiscrete topology Ind on a set with at least two elements is never T_0.
(2) consider $\mathbb{R}$ with the euclidean topology E. The topological space $(\mathbb{R}, \mathsf{E}) \times (\mathbb{R}, \mathsf{Ind})$ is not T_0, in fact (a, b) and (a, c) are not distinguishable.

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[1] i.e. $x \in \mathrm{cl}(y)$ and viceversa.

1.2. **T_1.** There is no standard name to refer to T_1-spaces. Some people call them **accessible**, or **Tychonoff,** or **Fréchet.** I suggest not to use these names, because they are not very well established, especially the last one might be confused with other terminologies coming from functional analysis.

Exercise 3. A space is T_1 if and only if points are closed sets.

Exercise 4. T_1-spaces are T_0.

Exercise 5. T_1-spaces are stable under products but not under subspaces.

Exercise 6. The following is a list of **non** T_1 spaces.

(1) The Sierpiński space S (the only topology on a set with two points where one is open and the other is closed) is not T_1.
(2) The Zariski topology on a commutative ring is not in general T_1.

1.3. **Hausdorff, seu T_2.**

Exercise 7. A space $\mathcal{X}$ is T_2 if and only if the diagonal $\{(x,x)\}_{x\in\mathcal{X}}$ is a closed set in the topological product $\mathcal{X}\times\mathcal{X}$.

Exercise 8. T_2-spaces are stable under product and subspaces.

T_2-spaces are separated *enough*, in the sense that points retain some information about the topology. We will see two examples of this behaviour, the first one is the exercise below, the second one is Prop. 1.1, where we show that $\mathbb{Q}$ knows everything about continuous functions defined on $\mathbb{R}$.

Exercise 9 (Uniqueness of limits in T_2-spaces)**.** Let $\mathcal{X}$ be a T_2-space and $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in $\mathcal{X}$. Then there is at most one point x that does not belong to and $\{x_n\}_{n\in\mathbb{N}}$ but is near it.

Exercise 10. Let $f,g : \mathcal{X} \rightrightarrows \mathcal{Y}$ be two continuous functions and $\mathcal{Y}$ be a T_2-space. Then the subspace of $\mathsf{E}(f,g)\subset\mathcal{X}$ defined by

$$\mathsf{E}(f,g) \coloneqq \{x\in\mathcal{X} : f(x)=g(x)\},$$

is a closed subset of $\mathcal{X}$.

Proof. Consider the continuous function $f\times g : \mathcal{X}\to\mathcal{Y}^2$ mapping $x\mapsto(f(x),g(x))$. Since $\mathcal{Y}$ is T_2, the diagonal $\Delta_{\mathcal{Y}}$ is closed in the product $\mathcal{Y}^2$. To finish, observe that

$$\mathsf{E}(f,g) = (f\times g)^{-1}\Delta_{\mathcal{Y}}$$

and thus must be closed because $f\times g$ is continuous. □

Definition 3. A subset D of a topological space is dense if its closure is the whole space.

Proposition 1.1. If D is a dense subset of $\mathcal{X}$ and $\mathcal{Y}$ is a Hausdorff space, then every continuous function $f : D\to\mathcal{Y}$ extends in at most one way to a continuous function from $\mathcal{X}$ to $\mathcal{Y}$.

Proof. Consider two extensions $g,h : \mathcal{X}\to\mathcal{Y}$. $\mathsf{E}(g,h)$ is a closed set containing D, thus it contains its closure, that is the whole space. This proves that $\mathcal{X}\subset\mathsf{E}(g,h)$, or equivalently that g coincides with h on $\mathcal{X}$. □

Example 4. The Zariski topology is not T_2.

1.4. **Regular Hausdorff and higher separation axioms.** I will not go very much into higher notion of separation, you should know that one can go at least as far as T_6, passing trough $\mathsf{T}_{3^1/_2}$. I shall say something on the notion of T_3-spaces, also known as *regular*. Regularity is the *correct* notion to study abstractly metrizability.

Example 5. Metric spaces are regular.

Theorem 1.2 (Uryshon)**.** A space $\mathcal{X}$ is metrizable if and only if it is regular and *second countable*.

2. Exercises

Pay attention, exercises labelled by the tea cup ☕ may not be incredibly challenging, even not challenging, but it is important keep them in mind, so take your time when solving them and be careful to find a formal and correct solution. Exercises labelled by the danger international sign ⚠ are very challenging. Exercise labelled by 📖 come from the beautiful book **Elementary Topology Problem Textbook**, by Viro, Ivanov, Netsvetaev and Kharlamov.

Remark 6. Special rule for this week, **solve at least 3 exercises**!

Exercise 11 (☕). Let S be the Sierpinski space and let $\mathcal{X} = (x, \chi)$ be any T_1-space. Prove that there is an embedding $e : \mathcal{X} \to S^{\mathcal{X}}$.

Exercise 12. Let C be a closed subset of the T_3 space $\mathcal{X}$. Let $\sim$ be the equivalence relation on $\mathcal{X}$ defined by

$$x \sim y \text{ iff } x = y \text{ or } \{x, y\} \subset A.$$

Let $\mathcal{X}/A$ denote the quotient space $\mathcal{X}/\sim$. Prove that $\mathcal{X}/A$ is Hausdorff.

Exercise 13. Let $f : \mathcal{X} \to \mathcal{Y}$ be a continuous function, where $\mathcal{Y}$ is Hausdorff. Prove that the *graph*

$$\Gamma(f) := \{(x, f(x)) \in \mathcal{X} \times \mathcal{Y}\}$$

is closed in $\mathcal{X} \times \mathcal{Y}$.

Exercise 14. Prove that the set of fixed points of a continuous map from a Hausdorff space to itself is a closed set.