TOPOLOGY

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ABSTRACT. This note summarizes the content of the fourth lesson of tutoring on the course Topology 2019. Also, attached at the end, there is an exercise sheet.

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1. COMPACTNESS

1.1. A vague intuition.

Remark 1. Sometimes people say that a good intuition on the notion of compactness is finiteness, compact spaces are to spaces like finite sets are to sets. This analogy is based on both a very geometric perspective on finite sets and some empirical evidences.

Exercise 1. A subset F of *the real world* ($\mathbb{R}^3$) is finite if and only if the subspace topology is discrete and compact.

Proof. If F is finite, then the subspace topology is discrete, in fact let ρ be the minimum of the distances between the points $p_i \in P$, then $B(p_i, \frac{\rho}{2}) \cap F = \{p_i\}$. Each of these sets is open in the topology of subspace, thus every point is open, so the induced topology is discrete. This space is trivially compact, because the number of open sets is finite in first place. Now, assume that F is discrete and compact, then every point is open, and thus the family of points is an open cover of F. Since F is compact we can extract a finite subfamily (of points) that covers F. This means that F was finite in first place. □

Remark 2. This is a very interest mathematical statement, in the sense that is assumes that every concept is inherently geometric, when we discuss about *mere* sets, we are in fact talking about geometric entities of which we are ignoring their topological status. The previous lemma proves that from this geometric point of view, **finite** is facile understanding of **discrete** *and* **compact**.

Remark 3. As a result of the previous remark, finite is at most a superficial shorthand for the notion of compactness, yet some characteristic properties of being finite have to reflect on compact spaces. We shall list some properties of finite sets that naturally come to mind to see if they are shared with compact spaces:

(1) a subset of a finite set is finite;
(2) the image of a finite set is a finite set;
(3) every function from a finite set $X \to \mathbb{R}$ has a maximum;

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In the following lemmas we see how this lemmas generalize to compact spaces.

Exercise 2. A closed subset of a compact space is compact.

Proof. Skipped. □

Exercise 3. A continuous function $f : \mathcal{X} \to \mathcal{Y}$ maps compacts into compacts.

Proof. Let C be a compact subspace. Let U_i be an open cover of $f(C)$, then $f^{-1}(U_i)$ is an open cover of C. Since C is compact, we can extract a finite family $f^{-1}U_1, \dots, f^{-1}U_n$ of open sets covering C. Thus $U_1, \dots, U_n$ covers $f(C)$, as desired. □

Exercise 4. Every continuous function from a compact $\mathcal{X}$ space to $\mathbb{R}$ has a maximum.

Proof. Let $f : \mathcal{X} \to \mathbb{R}$ be a function, $f(\mathcal{X})$ is compact in $\mathbb{R}$, thus it is closed and limited. This precisely means that f is bounded and that its supremum is reached. □

Remark 4. I hope you catched the motto here: there is some trace of compactness in the notion of finiteness and we can use our intuition on finiteness to guess the behaviour of compact spaces.

1.2. A *non metric* example of compact space.

Remark 5. It is possible to reformulate the notion of compactness in term of closed sets. A space $\mathcal{X}$ is compact if and only if, given a family of closed sets C_i such that $\bigcap C_i = \emptyset$, there is a finite subfamily whose intersection is empty. Of course this formulation is obtained taking the complement of the formulation with open sets.

Remark 6. The most natural example of compact space is a closed bounded subset of a metric spaces. On the other hand, those are not the only example of compact spaces, some very interesting spaces are compact!

Example 7. $\mathbb{R}$ with the cofinite topology is compact.

Proof. Let C_i be a family of closed sets whose intersection is empty. Recall that in this topology a set is closed if and only if is finite. Consider C_1, and call $p_1, \dots, p_k$ its points. Since the intersection of the whole family is empty, there exists C_j such that $p_j \notin C_j$. Thus this selection of C_j's together with C_1 has empty intersection, as desired. □

1.3. Toolbox and puzzles.

Exercise 5. Let $f : \mathcal{X} \to \mathcal{Y}$ be a continuous map, where $\mathcal{X}$ is compact and $\mathcal{Y}$ is Hausdorff. Then f is closed.

Proof. Proof. Let Z be a closed subset of $\mathcal{X}$. We know that it must be compact too. Images of compact spaces under continuous maps are again compact, so$f(Z)$ is compact. But that means it is also closed, because Y is Hausdorff. □

Exercise 6. Let $\mathcal{Y}$ be a compact space, then the projection $\pi : \mathcal{X} \times \mathcal{Y} \to \mathcal{X}$ is closed.

Proof. Try yourself! □

Exercise 7 (Characterization of continuous functions for compact Hausdorf spaces)**.** Let $f : \mathcal{X} \to \mathcal{Y}$ be a continuous map with $\mathcal{Y}$ compact Hausdorff. Then f is continuous if and only if the graph $\Gamma(f)$ is closed.

Proof. Assume that f is continuous, we have already seen that if $\mathcal{Y}$ is Hausdorff, the graph is closed, thus in this case the graph is closed *a fortiori*. For the other implication, assume that the graph is closed. Let C be an closed set in $\mathcal{Y}$. Then the intersection $\Gamma(f) \cap (\mathcal{X} \times C)$ is also closed. If we apply Exercise 6, we can deduce that

$$\pi(\Gamma(f) \cap (\mathcal{X} \times C)) = f^{-1}(C)$$

is closed and equivalently, that f is continuous.

□

2. Exercises

Pay attention, exercises labelled by the tea cup ☕ may not be incredibly challenging, even not challenging, but it is important keep them in mind, so take your time when solving them and be careful to find a formal and correct solution. Exercises labelled by the danger international sign ⚠ are very challenging. Exercise labelled by 📘 come from the beautiful book **Elementary Topology Problem Textbook**, by Viro, Ivanov, Netsvetaev and Kharlamov.

Exercise 8 (☕). Prove Ex. 6.

The Book (📘). We dig into the chapter about compactness.

17'1
17'3
17'4
17'9 X,Y,Z.