TOPOLOGY

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ABSTRACT. This note summarizes the content of the fifth lesson of tutoring on the course Topology 2019. Also, attached at the end, there is an exercise sheet.

CONTENTS

1. CONNECTEDNESS

Hello, and welcome to the fifth lesson of this course in topology. The lesson of today is mainly based on **the book** (📕). In the last lesson I tried to give you an intuitive description of compactness. In the case of connectedness it is very hard to give a better description than the one that the nature provided us with. A space is connected, if it is connected in the colloquial sense of the word. On the other hand, be careful, for humans it is quite easy to confuse connectedness with the very visual notion of *path connectedness*!

I chose three topic where we can see how to **apply connectedness** as a **technical tool** in order to prove some statements. We will try to be very interactive in this lesson, thus the proofs are not provided in the notes.

1.1. Intermediate Value Theorem and its Generalizations.

The following theorem is usually included in Calculus. In fact, in a sense it is equivalent to connectedness of the segment.

Exercise 1 (Intermediate Value Theorem)**.** A continuous function $f : [a, b] \to \mathbb{R}$ takes every value between $f(a)$ and $f(b)$.

Remark 1. Many problems that can be solved by using the Intermediate Value Theorem can be found in Calculus textbooks. Here are few of them.

Exercise 2. Any polynomial of odd degree in one variable with real coefficients has at least one real root.

Exercise 3. Let $\mathcal{X}$ be a connected space, $f : \mathcal{X} \to \mathbb{R}$ be a continuous function. Then $f(\mathcal{X})$ is an interval of $\mathbb{R}$.

Exercise 4. Let $J \subset \mathbb{R}$ be an interval of the real line and $f : J \to \mathbb{R}$ be a continuous function. Then $f(J)$ is also an interval of $\mathbb{R}$.[1]

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[1] In other words, continuous functions map intervals to intervals.

1.2. **Applications to Homeomorphism Problem.** Connectedness is a topological property, and the number of connected components is a topological invariant. Simple constructions assigning homeomorphic spaces to homeomorphic ones (e.g., deleting one or several points), allow us to use connectedness for proving that some connected spaces are not homeomorphic.

Exercise 5. $[0,2]$ and $[0,1] \cup [2,3]$ are not homeomorphic.

Exercise 6. $[0,\infty)$, $\mathbb{R}$, and S^1 are pairwise nonhomeomorphic.

Exercise 7. A circle is not homeomorphic to a subspace of $\mathbb{R}$.

Exercise 8. That square and segment are not homeomorphic.

Exercise 9. $\mathbb{R}$ and $\mathbb{R}^n$ are not homeomorphic if $n > 1$.

1.3. **Induction on Connectedness.** Here we see a typical proof technique in topology.

Definition 2. A map $f : \mathcal{X} \to \mathcal{Y}$ is locally constant if each point of X has a neighborhood U such that the restriction of f to U is constant.

Exercise 10. Any locally constant map is continuous.

Exercise 11. A locally constant map on a connected set is constant.

Exercise 12 (Induction on Connectedness)**.** Let E be a property of subsets of a topological space $\mathcal{X}$ such that the union of sets with nonempty pairwise intersections inherits this property from the sets involved. If $\mathcal{X}$ is connected and each point in $\mathcal{X}$ has a neighborhood with property E, then $\mathcal{X}$ also has property E.

2. EXERCISES

Pay attention, exercises labelled by the tea cup ☕ may not be incredibly challenging, even not challenging, but it is important keep them in mind, so take your time when solving them and be careful to find a formal and correct solution. Exercises labelled by the danger international sign ⚠ are very challenging. Exercise labelled by 📕 come from the beautiful book **Elementary Topology Problem Textbook**, by Viro, Ivanov, Netsvetaev and Kharlamov.

Remark 3. Special rule for this week: **Pick something juice from each of the three exercises**.

The Book (📕). We dig into the chapter about connectedness.

12’4 G,H,J,K.
13’4x Everything except the riddles.
14’9 14.27x.(1)