TOPOLOGY

IVAN DI LIBERTI

ABSTRACT. This note summarizes the content of the sixth lesson of tutoring on the course Topology 2019. Also, attached at the end, there is an exercise sheet.

CONTENTS

1. BONUS TRACK: TOPOLOGICAL GROUPS

1.1. **Motivations and definitions.** Hello and welcome to this lesson of topology. In the previous episodes of this show we studied some properties of topological spaces: separation axioms, connectedness, compactness. Those are intended to be both technical tools to prove statements and conceptual properties that a space may or may not have. Today we study topological groups. For us they'll be an excuse to apply what we learnt in the previous lessons, but a matter of fact, topological groups are a research field per se and a very interesting object of study. A topological group is a group that is also a topological space, where the group structure interact with the topology nicely. Before giving the precise definition, we start with two motivating example.

Example 1. Besides its metric properties, $\mathbb{R}$ is in first place a field, and even more elementarily an abelian group together with the sum. The structure of field and topological space are designed to interact, for example $\_+\_ : \mathbb{R}^2 \to \mathbb{R}$ is a continuous function with respect to the metric topology. The fact that

$$\lim_{(a_n,b_n)\to(a,b)} a_n + b_n = a + b$$

will not surprise anybody.

Example 2. $\mathbb{C}^*$, the multiplicative group of the field $\mathbb{C}$ has a natural topological structure inherited from being an open subset of $\mathbb{R}^2$. Also the multiplication is continuous with respect to the metric topology induced by $\mathbb{R}^2$, in fact the product of complex number $(a + \mathrm{i}b)\cdot(c + \mathrm{i}d) = (ac - bd) + \mathrm{i}(ad + bc)$ results in a polinomial function $\mathbb{R}^4 \to \mathbb{R}^2$ assigning

$$(a, b, c, d) \mapsto (ac - bd, ad + bc)$$

that is clearly continuous.

Definition 3. A topological group $\mathcal{G} = (G, \gamma, \circ, 1, \_^{-1})$ is a topological space (G, γ) together with a group structure $(G, \circ, 1, \_^{-1})$ such that the functions $\circ$ and $\_^{-1}$ are continuous with respect to γ.

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Definition 4. A morphism of topological groups is a group homomorphism that is also a continuous map.

Example 5. Let $S^1 \subset \mathbb{C}$ the subset of complex numbers such that $z\bar{z} = 1$. From a topological point of view this is precisely the circle. Being a subset of $\mathbb{C}$ that does not contain the 0, it is legitimate to wonder if it is a subgroup of $\mathbb{C}^*$, the multiplicative group of the field $\mathbb{C}$. Of course the answer is affirmative, in fact 1 has norm 1 and if z, w have norm 1, their product have norm 1,

$$\bar{z}\bar{w}zw = \bar{z}z\bar{w}w = 1 \cdot 1 = 1.$$

Thus $S^1 < \mathbb{C}^*$ is a topological subgroup and the inclusion is a morphism of topological groups.

Example 6. The general linear group $\mathsf{GL}(\mathbb{R}^n)$ inherits a natural topology from being an open subsets of $\mathsf{End}(\mathbb{R}^n) \cong \mathbb{R}^{n^2}$. It is open because the determinant $\mathsf{det} : \mathsf{End}(\mathbb{R}^n) \to \mathbb{R}$ is a continuous function and $\mathsf{GL}(\mathbb{R}^n)$ coincides with $\mathsf{det}^{-1}(\{x \neq 0\})$. The multiplication of matrices in polynomial in every entry, and thus is globally a continuous function $\mathbb{R}^{n^2} \times \mathbb{R}^{n^2} \to \mathbb{R}^{n^2}$. Using the adjugate matrix, one can notice that also the inverse matrix M^{-1} is a rational function of the entries of M, thus $\mathsf{GL}(\mathbb{R}^n)$ is a topological group. Observe that this argument proves also that $\mathsf{End}(\mathbb{R}^n)$ is a topological monoid.

Remark 7. The interaction between the topological structure and the algebraic structure makes a topological group a quite rigid object. As soon as some weak property is verified, the algebraic structure rigidifies it. We will see two examples of this behaviour.

1.2. **Topological groups and separation axioms.**

Lemma 1.1. A topological group is homogeneous, that is, given two points there is an automorphism of topological spaces swapping them.

Proof. Let g, h be two elements in a topological group $\mathcal{G}$. The map $hg^{-1} : \mathcal{G} \to \mathcal{G}$ mapping $x \mapsto hg^{-1}x$ is a continuous because it can be obtained via the composition

$$\mathcal{G} \xrightarrow{\mathrm{id}\times hg^{-1}} \mathcal{G} \times \mathcal{G} \xrightarrow{\circ} \mathcal{G},$$

that is clearly a composition of continuous maps. Now observe that

$$hg^{-1}(g) = hg^{-1}g = h.$$

□

Remark 8. Be careful, that is not an automorphism of topological groups! It is not a group homomorphism!

Lemma 1.2. A topological group is T_1 if and only if the identity e is a closed point.

Proof. One implication is trivial, if the space is T_1 every point is closed. For the other implication observe that since the space is homogeneous, as soon as a point is closed, every point is closed, because homeomorphism are closed maps. □

Lemma 1.3. A topological group is T_1 if and only if it is T_2.

Proof. One implication is trivial, in fact if a space is T_2, it is T_1. For the other implication, we will show that the diagonal Δ is closed in the product $\mathcal{G}\times\mathcal{G}$. Recall that the composition $\circ : \mathcal{G} \times \mathcal{G} \to \mathcal{G}$ is continuous, since the space is T_1, the unit e is closed and thus

$$\Gamma = \{(g, g^{-1})\} = \circ^{-1}(e)$$

is closed. If we prove that there is a topological automorphism of $\mathcal{G}$ swapping Δ and Γ we are done, because it will send closed sets into closed sets. In fact this is quite easy to provide and is the map $\mathcal{G} \times \mathcal{G} \xrightarrow{\mathrm{id}\times\_^{-1}} \mathcal{G} \times \mathcal{G}$. □

1.3. **Topological groups and connectedness.**

Lemma 1.4. Let $\mathcal{H} < \mathcal{G}$ be a normal and discrete subgroup of a path connected topological group $\mathcal{G}$. Then $\mathcal{H}$ is in the center of $\mathcal{G}$.

Proof. We shall prove that for every $g \in \mathcal{G}$ and every $h \in \mathcal{H}$ one has that $gh = hg$. Since $\mathcal{G}$ is path connected there is a path $\gamma : [0,1] \to \mathcal{G}$ connecting the identity e with g. Now define

$$\gamma_h : t \mapsto \gamma(t)h\gamma(t)^{-1}.$$

Since $\mathcal{H}$ is normal, this is a continuous map $\gamma_h : [0,1] \to \mathcal{H}$. Since $\mathcal{H}$ is discrete and $[0,1]$ is connected, it must be a constant map. Since $\gamma_h(0) = ehe = h$, we can deduce that γ_h is the constant map h. But observe that $\gamma_h(1)$ is ghg^{-1}, thus $ghg^{-1} = h$, or equivalently $gh = hg$. □

2. EXERCISES

Pay attention, exercises labelled by the tea cup may not be incredibly challenging, even not challenging, but it is important keep them in mind, so take your time when solving them and be careful to find a formal and correct solution. Exercises labelled by the danger international sign are very challenging. Exercise labelled by come from the beautiful book **Elementary Topology Problem Textbook**, by Viro, Ivanov, Netsvetaev and Kharlamov.

Remark 9. No exercises for this week. Have fun.