TOPOLOGY

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ABSTRACT. This note summarizes the content of the seventh lesson of tutoring on the course Topology 2019. Also, attached at the end, there is an exercise sheet.

CONTENTS

1. WRITING DOWN MAPS

Topologists love to wave their hands and say: *That's the proof*. They *glue, cut, cross, move continuously*, and so on. Is this a sound way of doing mathematics?! Maybe it shouldn't be but as a matter of fact, it is. There exists an informal dictionary between their sign language and formal mathematics. The aim of this lesson is to see some standard methods about how to write down simple maps. Eventually you will not write none of these maps, as any professional topologist, you will say: *That's completely evident*, and move your hands the way the proof would right down.

1.1. A non closed map.

We saw that a continuous map from a compact space into a Hausdorff space is always closed, it is useful to see at least a non closed map. There are many, very simple, examples. My favourite is the projection $\pi_x : \mathbb{R}^2 \to \mathbb{R}$. In fact, consider the map $\phi : \mathbb{R}^2 \to \mathbb{R}$ that takes:

$$(x, y) \mapsto x \cdot y.$$

This map is continous, thus $\phi^{-1}(1)$ is closed in $\mathbb{R}^2$. On the other hand its projection onto the first coordinate is $\mathbb{R} \setminus \{0\}$ which is not closed because it's open and $\mathbb{R}$ is connected.

1.2. Writing down homeomprhisms.

Example 1. Sometimes it is very easy to prove that two spaces are homeomorphic, that is the case of $(0, 1)$ and $(0, 5)$ where it is enough to map $t \mapsto 5t$ in order to get a homeorphism. In some other cases, even if it is quite easy, one has to find the map that works, as in the case of $(0, 1)$ and $\mathbb{R}$. A modified version of this last example is one of the exercises for the next time.

Date: 1 April 2019.

1.2.1. *Two models of the circle.* The circle S^1 in in general defined as the subset of $\mathbb{R}^2$ of those points whose norm is 1. On the other hand it is quite useful to give at least another presentation of this object by the quite standard process of *gluing*. Let $\mathtt{I}$ be the segment $[0,1]$ and $\sim$ be the equivalence relation that collapses 1 and 0 to the same point. We claim that the quotient space $\circ$ is homeomorphic to S^1.

Proof. Consider the map $\sigma : [0,1] \to \mathbb{R}^2$ mapping

$$t \mapsto (\cos(2\pi t), -\sin(2\pi t)).$$

Obviously $|\sigma(t)| = \cos^2(t)+\sin^2(t) = 1$, so no doubt that the image of this map is contained in the circle. Moreover, given a point $(x, y) \in \mathsf{S}^1$ we have that

$$(x, y) = \sigma(\arctan(y/x)),$$

that means that σ is surjective (on the circle). σ is clearly continuous and it is very easy to check that it is injective on the quotient space $\circ$. Thus, $\sigma : \circ \to \mathsf{S}^1$ is continuous, surjective and and injective. To prove that it is a homemorphism, we should show that the inverse is continuous, or equivalently that σ is open. Instead we show that σ is closed, and injective closed map is always open. σ closed because it's a map from a compact space into an Hausdorff space. □

1.2.2. *The projective line.* Consider the topological space $\mathbb{R}^2 \setminus \{0\}$ as an open subspace of $\mathbb{R}^2$ with the usual metric topology. Define the equivalence relation that identifies two vectors v, w if and only if there is a real non 0 number λ such that $v = \lambda w$. The resulting quotient space is called *real projective line* and is indicated by $\mathbb{RP}^1$. With the same idea in mind one can define $\mathbb{RP}^n$.

Exercise 1. $\mathbb{RP}^1 \cong \mathsf{S}^1$.

Proof. We start from the combinatorial presentation of the circle: $\circ$ obtained in the previous exercise. We shall find a homeomorphism $\circ \to \mathbb{RP}^1$. As in the previous case it is enough to find a continuous, surjective and injective map, because it will be trivially closed. Define

$$\phi : t \mapsto (\cos(4\pi t), -\sin(4\pi t)).$$

ϕ is defined to be a map from $\mathtt{I} \to \mathbb{R}^2 \setminus \{0\}$. When we compose it with the projection $\mathbb{R}^2 \setminus \{0\} \to \mathbb{RP}^1$ it is evidently surjective, in fact every element v is equivalent to $\frac{v}{|v|}$ in the quotient. The maps is also injective and continuous and it is closed because the domain is compact and the codomain is Hausdorff. □

1.3. **Writing down homotopies.** For the following section I recommend the reader to draw as much as possible every geometric object and map that I present.

1.3.1. $\mathbb{R}^n$ *is a point.* The two spaces are homotopy equivalent. Recall that this means that there are two maps

$$p : \cdot \leftrightarrows \mathbb{R}^n : t$$

such that tp is equivalent to the identity of the point and tp is equivalent to the identity of $\mathbb{R}^n$. t is forced to be the constant map. p might be any map, for this exercise we choose the map that sends the point to 0. tp is precisely the identity of the point, and thus is clearly equivalent to the identity. pt instead is the map that send every point to 0. The homotopy between this map and the identity is the map $\mathcal{H} : \mathbb{R}^n \times I \to \mathbb{R}^n$ assigning:

$$(v, t) \mapsto (1 - t) \cdot v.$$

When t is 0, $\mathcal{H}(0, \_)$ coincides with the identity, while when t is 1, $\mathcal{H}(1, \_)$ is precisely the constant map sending everything to 0.

1.3.2. $\mathbb{R}^n \setminus 0$ *is* S^{n-1}. As in the previous example, we are looking for a couple of maps

$$\iota : \mathsf{S}^{n-1} \leftrightarrows \mathbb{R}^n \setminus 0 : \frac{\_}{n(\_)}.$$

The map on the left is simply the inclusion, the map on the right assigns $v \mapsto \frac{v}{|v|}$. Observe that the latter is well defined because we removed the origin from $\mathbb{R}^n$. As in the previous exercise the composition that lands in the sphere is the identity, and thus it is homotopically equivalent to the identity. The other composition is instead is equivalent to the identity via the map $\mathcal{H} : \mathbb{R}^n \setminus \{0\} \times I \to \mathbb{R}^n \setminus \{0\}$ assigning:

$$(v, t) \mapsto (1 - t) \cdot v + t \cdot \frac{v}{|v|}.$$

When t is 0, $\mathcal{H}(0, \_)$ coincides with the identity, while when t is 1, $\mathcal{H}(1, \_)$ is precisely the constant map sending v to $\frac{v}{|v|}$.

2. Exercises

Pay attention, exercises labelled by the tea cup ☕ may not be incredibly challenging, even not challenging, but it is important keep them in mind, so take your time when solving them and be careful to find a formal and correct solution. Exercises labelled by the danger international sign ⚠ are very challenging. Exercise labelled by 📕 come from the beautiful book **Elementary Topology Problem Textbook**, by Viro, Ivanov, Netsvetaev and Kharlamov.

Remark 2. All the exercises are mandatory.

Exercise 2 (Models of the torus)**.** Let $\square$ be $[0, 1] \times [0, 1]$ and $\sim$ be the equivalence relation that identifies opposite points of the border. Then the space $T = \square / \sim$ is homeomorphism to $\mathsf{S}^1 \times \mathsf{S}^1$.

Exercise 3 (The square and the circle)**.** Prove that the circle is homeomorphic to the border of the $\square$.

Exercise 4 (The interval and the whole line)**.** Find an homemorphism between $(5, 7)$ and $\mathbb{R}$.