TOPOLOGY

IVAN DI LIBERTI

ABSTRACT. This note summarizes the content of the 8th lesson of tutoring on the course Topology 2019. Also, attached at the end, there is an exercise sheet.

CONTENTS

1. THE BIG PICTURE

Hello, this is the first of a trilogy of lectures. Whe shall try to challenge the following questions:

(1) Given a topological space $\mathcal{X}$ is there an efficient way to compute its fundamental group $\pi_1(\mathcal{X})$?
(2) In particular, is there a honest and algebraic based way to prove that $\pi_1(\mathsf{S}^1) = \mathbb{Z}$?
(3) Given a group G is there a space $\mathcal{X}_G$ whose fundamental group is isomorphic to G, i.e. $\pi_1(\mathcal{X}_G) \cong G$.?

1.1. **Motivations.**

1.1.1. *Computing* $\pi_1(\mathcal{X})$. The fundamental group is a topological invariant, and thus it is a suitable tool to discriminate whether two spaces are (not) homeomorphic or homotopically equivalent. For example, one can use the fundamental group to prove that $\mathbb{R}^2 \setminus \{0\}$ is not homeomorphic, nor homotopy equivalent to $\mathbb{R}^2$. For this reason, having good strategies to compute the fundamental group is useful in order to classify spaces from both, the homotopical and the topological, points of view.

1.1.2. $\pi_1(\mathsf{S}^1) = \mathbb{Z}$. The reason behind the second question is twofold. On one hand, in this course we never saw a formal proof of this result, on the other we want to see how far one can push the algebraic approach to solving topological problems.

Date: 8 April 2019.

1.1.3. $\pi_1(\mathcal{X}_G) \cong G$. This question comes to mind quite naturally in a very speculative way. The class of those groups that can be obtained as fundamental groups might be very special in principle, maybe there is some hidden additional (algebraic) structure that a fundamental group always carries, maybe not. Solving this inverse problem, especially if the answer is that is class is very special might teach something about our perception of loops. In this case the answer to this question will come a very simple side effect of the previous questions, and thus we decided to include it in our tour.

1.2. **Our strategy.**

- 8^{th} Bestiary II: Actions, projective spaces and the circle;
- 9^{th} Artillery: Properly discontinuous actions and coverings;
- 10^{th} Theorems: Computing π_1 via coverings.

1.2.1. *Bestiary.* In this lecture we will introduce some relevant class of spaces of which it would be interesting, for some reasons, to compute the fundamental group. This should be seen as the natural prosection of our second lecture. The examples in that case were coming from general topology and were of any sort. In this case the example are very geometric and would interest the *true* topologist.

1.2.2. *Artillery.* This lecture per se might look very technical. We will present our main tool to compute fundamental groups. This is the tip of the iceberg of the rich and beautiful theory of covering spaces. Covering spaces were introduced in complex analysis and are also connected to the problem of integrating vector fields, that was in fact somehow, their first use.

1.2.3. *Theorems.* We will show precisely how to use the artillery in order to answer our initial questions. All of them will find answer in this lesson.

2. Bestiary II: Actions, projective spaces and the circle

2.1. **Projective spaces.** In the previous lecture we introduced the projective line $\mathbb{RP}^1$ as a quotient of $\mathbb{R}^2 \setminus \{0\}$. This construction can be generalized for every n, obtaining the projective space $\mathbb{RP}^n$. These spaces are quite relevant in *projective geometry* and *affine algebraic geometry*. The construction is the following. Consider the space $\mathbb{R}^{n+1} \setminus \{0\}$ with the usual topology and equip it with the equivalence relation $\sim$ that identifies two vectors $v \sim w$ if and only if there is an invertible number $\lambda \in \mathbb{R}^*$ such that $v = \lambda w$.

Definition 1. The **projective space** $\mathbb{RP}^n$ of dimension n is defined as the quotient of $\mathbb{R}^{n+1} \setminus \{0\}$ under the previous equivalence relation.

Remark 2. The same definition can be given for complex vector spaces, obtaining $\mathbb{RC}^n$.

Remark 3 (Yet another presentation)**.** Consider the inclusion of the sphere $\iota : \mathsf{S}^n \hookrightarrow \mathbb{R}^{n+1} \setminus \{0\}$. Then the composition $\mu : \mathsf{S}^n \hookrightarrow \mathbb{R}^{n+1} \setminus \{0\} \twoheadrightarrow \mathbb{RP}^n$ is surjective, in fact every vector v is equivalent to its normalization $\frac{v}{|v|}$. The fibre of a point is

$$\mu^{-1}([p]) = \{p, -p\}.$$

Thus $\mathbb{RP}^n$ can be obtained as a quotient of the n-sphere under the equivalence relation that identifies every point p to its opposite $-p$.

2.2. **Actions.**

Definition 4. Let $\mathcal{X}$ be a topological space and G be a group. A **continuous action**[1] $\chi : G \curvearrowright \mathcal{X}$ is a group homomorphism $\chi : G \to \text{Homeo}(\mathcal{X})$. Equivalently, it is a group action of G on the underlying set X such that $\chi(g)$ is a homeomorphism for every $g \in G$.

[1] Just action for short.

Remark 5. Recall that the data of an action is the same of a map $\cdot : G \times \mathcal{X} \to \mathcal{X}$ such that

$$(gh) \cdot x = g \cdot (h \cdot x)$$
$$e \cdot x = x.$$

Remark 6. Let $\chi : G \curvearrowright \mathcal{X}$ be a continuous action of a group. Then it is possible to define an equivalence relation on $\mathcal{X}$ as follows. We say that $x \sim_\chi y$ if and only if there exist an element $g \in G$ such that $x = g \cdot y$. The quotient space is very often indicated with $\mathcal{X}/G$.

Remark 7 (Projective spaces)**.** The projective space $\mathbb{RP}^n$ can be obtained as the quotient of S^n under the action of a group. In fact consider the cyclic group of order two $\mathbb{Z}_2$ and its action $\mathbb{Z}_2 \curvearrowright \mathsf{S}^n$ on S^n given by

$$[n] \cdot p = (-1)^n p.$$

The induced equivalent relation is precisely the same of the Rem 3 and thus the obtained quotient space is $\mathbb{RP}^n$.

Example 8 (The circle)**.** The circle S^1 can be obtained as the quotient of of $\mathbb{R}$ under the action of the integers $\mathbb{Z}$. The action $\mathbb{Z} \curvearrowright \mathbb{R}$ is the most natural in the world:

$$n \cdot x = x + n.$$

In order to prove that $\mathbb{R}/\mathbb{Z} \cong \mathsf{S}^1$, call $\iota : [0, 1] \subset \mathbb{R}$ the inclusion of the interval in the line. Now, the composition $[0, 1] \subset \mathbb{R} \to \mathbb{R}/\mathbb{Z}$ is surjective because every point x is equivalent to $x - [x] \leq 1$ and is injective on every point except for 0 and 1, that are both sent to the same element. This is precisely what we called the combinatorial presentation of the circle in the previous lesson.

3. Exercises

Pay attention, exercises labelled by the tea cup ☕ may not be incredibly challenging, even not challenging, but it is important keep them in mind, so take your time when solving them and be careful to find a formal and correct solution. Exercises labelled by the danger international sign ⚠ are very challenging. Exercise labelled by 📖 come from the beautiful book **Elementary Topology Problem Textbook**, by Viro, Ivanov, Netsvetaev and Kharlamov.

Remark 9. All the exercises are mandatory.

Exercise 1 (Models of projective plane)**.** Prove that $\mathbb{RP}^2$ is homeomorphic to the following quotient of the disk. Two points p, q in the disk D^2 are equivalent if and only if both have norm 1 and $p = -q$.

Exercise 2 (The square and the circle)**.** Find a nontrivial continuous action $\mathbb{Z}_n \curvearrowright \mathsf{S}^1$ of the cyclic group of order n on the circle and compute the quotient space.