§59 The Fundamental Group of S^n

Now we turn to a problem mentioned at the beginning of the chapter, the problem of showing that the sphere, torus, and double torus are surfaces that are topologically distinct. We begin with the sphere; we show that S^n is simply connected for $n \geq 2$. The crucial result we need is stated in the following theorem.

Theorem 59.1. *Suppose $X = U \cup V$, where U and V are open sets of X. Suppose that $U \cap V$ is path connected, and that $x_0 \in U \cap V$. Let i and j be the inclusion mappings of U and V, respectively, into X. Then the images of the induced homomorphisms*

$$i_* : \pi_1(U, x_0) \to \pi_1(X, x_0) \quad \text{and} \quad j_* : \pi_1(V, x_0) \to \pi_1(X, x_0)$$

generate $\pi_1(X, x_0)$.

Proof. This theorem states that, given any loop f in X based at x_0, it is path homotopic to a product of the form $(g_1 * (g_2 * (\cdots * g_n)))$, where each g_i is a loop in X based at x_0 that lies either in U or in V.

Step 1. We show there is a subdivision $a_0 < a_1 < \cdots < a_n$ of the unit interval such that $f(a_i) \in U \cap V$ and $f([a_{i-1}, a_i])$ is contained either in U or in V, for each i.

To begin, choose a subdivision $b_0, \ldots, b_m$ of $[0, 1]$ such that for each i, the set $f([b_{i-1}, b_i])$ is contained in either U or V. (Use the Lebesgue number lemma.) If $f(b_i)$ belongs to $U \cap V$ for each i, we are finished. If not, let i be an index such that $f(b_i) \notin U \cap V$. Each of the sets $f([b_{i-1}, b_i])$ and $f([b_i, b_{i+1}])$ lies either in U or in V. If $f(b_i) \in U$, then both of these sets must lie in U; and if $f(b_i) \in V$, both of them must lie in V. In either case, we may delete b_i, obtaining a new subdivision c_0, $\ldots$, c_{m-1} that still satisfies the condition that $f([c_{i-1}, c_i])$ is contained either in U or in V, for each i.

A finite number of repetitions of this process leads to the desired subdivision.

Step 2. We prove the theorem. Given f, let $a_0, \ldots, a_n$ be the subdivision constructed in Step 1. Define f_i to be the path in X that equals the positive linear map of $[0, 1]$ onto $[a_{i-1}, a_i]$ followed by f. Then f_i is a path that lies either in U or in V, and by Theorem 51.3,

$$[f] = [f_1] * [f_2] * \cdots * [f_n].$$

For each i, choose a path α_i in $U \cap V$ from x_0 to $f(a_i)$. (Here we use the fact that $U \cap V$ is path connected.) Since $f(a_0) = f(a_n) = x_0$, we can choose α_0 and α_n to be the constant path at x_0. See Figure 59.1.

Now we set

$$g_i = (\alpha_{i-1} * f_i) * \overline{\alpha_i}$$

for each i. Then g_i is a *loop* in X based at x_0 whose image lies either in U or in V. Direct computation shows that

$$[g_1] * [g_2] * \cdots * [g_n] = [f_1] * [f_2] * \cdots * [f_n]. \qquad \blacksquare$$

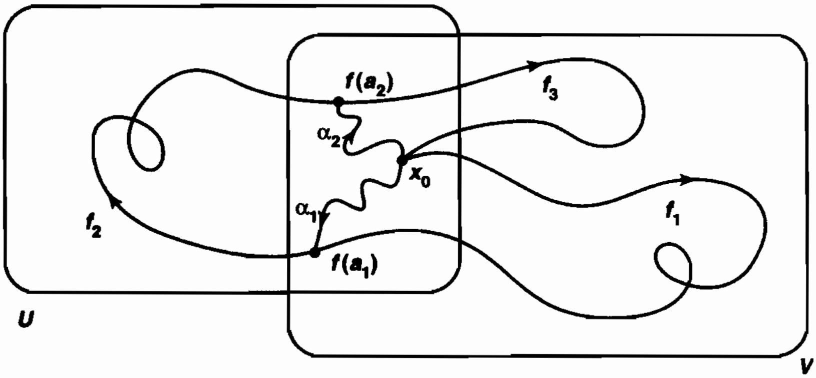

Figure 59.1

The preceding theorem is a special case of a famous theorem of topology called the *Seifert-van Kampen theorem*, which expresses the fundamental group of the space $X = U \cup V$ quite generally, when $U \cap V$ is path connected, in terms of the fundamental groups of U and V. We shall study this theorem in Chapter 11.

Corollary 59.2. *Suppose $X = U \cup V$, where U and V are open sets of X; suppose $U \cap V$ is nonempty and path connected. If U and V are simply connected, then X is simply connected.*

Theorem 59.3. *If $n \geq 2$, the n-sphere S^n is simply connected.*

Proof. Let $p = (0, \ldots, 0, 1) \in \mathbb{R}^{n+1}$ and $q = (0, \ldots, 0, -1)$ be the "north pole" and the "south pole" of S^n, respectively.

Step 1. We show that if $n \geq 1$, the *punctured sphere* $S^n - p$ is homeomorphic to $\mathbb{R}^n$.

Define $f : (S^n - p) \to \mathbb{R}^n$ by the equation

$$f(x) = f(x_1, \ldots, x_{n+1}) = \frac{1}{1 - x_{n+1}}(x_1, \ldots, x_n).$$

The map f is called ***stereographic projection***. (If one takes the straight line in $\mathbb{R}^{n+1}$ passing through the north pole p and the point x of $S^n - p$, then this line intersects the n-plane $\mathbb{R}^n \times 0 \subset \mathbb{R}^{n+1}$ in the point $f(x) \times 0$.) One checks that f is a homeomorphism by showing that the map $g : \mathbb{R}^n \to (S^n - p)$ given by

$$g(y) = g(y_1, \ldots, y_n) = (t(y) \cdot y_1, \ldots, t(y) \cdot y_n, 1 - t(y)),$$

where $t(y) = 2/(1 + \|y\|^2)$, is a right and left inverse for f.

Note that the reflection map $(x_1, \ldots, x_{n+1}) \to (x_1, \ldots, x_n, -x_{n+1})$ defines a homeomorphism of $S^n - p$ with $S^n - q$, so the latter is also homeomorphic to $\mathbb{R}^n$.