TOPOLOGY

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ABSTRACT. This note summarizes the content of the 10th lesson of tutoring on the course Topology 2019. Also, attached at the end, there is an exercise sheet.

1. USING COVERINGS

Recall that the main result of the previous lesson was a connection between groups acting on a simply connected space $G \curvearrowright \mathcal{E}$ and the fundamental group $\pi_1(\mathcal{E}/G)$ of the topological quotient.

Theorem 1.1 (Main theorem)**.** Let $G \curvearrowright \mathcal{E}$ be a properly discontinuous action of a group on a (path connected) simply connected space $\mathcal{E}$. Then $G \cong \pi_1(\mathcal{E}/G, x)$.

We used this result to provide a proof that the fundamental group of the circle is $\mathbb{Z}$, that represents the building block of the whole elementary algebraic topology.

Corollary 1.2. The fundamental group of the circle is $\mathbb{Z}$.

Remark 1. The proof was based on the observation that $\mathbb{Z}$ acts on the real line by translation. Then, since the real line is contractible (and thus is simply connected), we can apply the theorem and get the the fundamental group of the quotient space (the circle) is $\mathbb{Z}$.

Remark 2. This proof-line suggests a strategy: given a topological space we can use coverings to *blow up* some homotopical singularities and encode them in a action of a group over the unsingularized space (the covering). In this lesson we use this strategy to:

(1) compute the fundamental group of a projective space;
(2) compute and discuss the fundamental group of the topologist bag;

1.1. **The fundamental group of the projective space.**

Fact 1. The sphere S^n is simply connected.

Corollary 1.3. The fundamental group of a projective space $\mathbb{RP}^n$ is $\mathbb{Z}_2$, the cyclic group with two elements.

Proof. Recall that the topological data of $\mathbb{RP}^n$ can be presented as the quotient of the sphere S^n under the action of $\mathbb{Z}_2$,

$$[n] \cdot p = (-1)^n p.$$

Since the sphere is simply connected we can deduce that the fundamental group of $\mathbb{RP}^n$ is $\mathbb{Z}_2$. □

Remark 3. For the last time, we split the data of $\mathbb{RP}^n$ in two: $(\mathsf{S}^n, \mathbb{Z}_2)$. One is a homotopically simpler space, S^n, the other is a group. The homotopical information about $\mathbb{RP}^n$ is encoded in the action. This idea suggests that **a space is always the same as a simply connected covering together with an appropriate action of its fundamental group.**

Remark 4. Today we can't prove that the sphere S^n is simply connected, and in fact this will be precisely the content of the next lesson. *One should always leave the table a bit hungry.*

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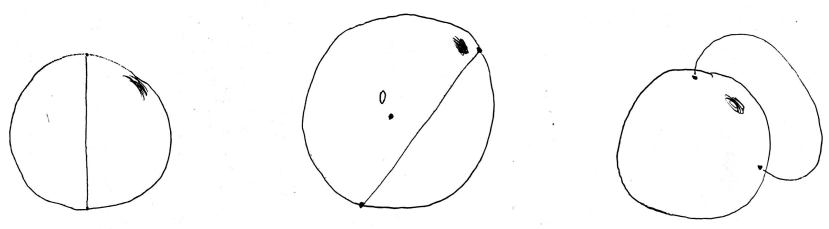
1.2. **The fundamental group of the topologist bag.**

Definition 5 (The topologist bag)**.** The topologist bag B is a subspace of $\mathbb{R}^3$ and is a sphere together with the diameter connecting the north pole to the south pole.

Exercise 1 (The topologist bag is a bag)**.** Prove that the topologist bag is homeomorphic to a sphere together with a path connecting the north pole to the south pole that does not lie inside the sphere.

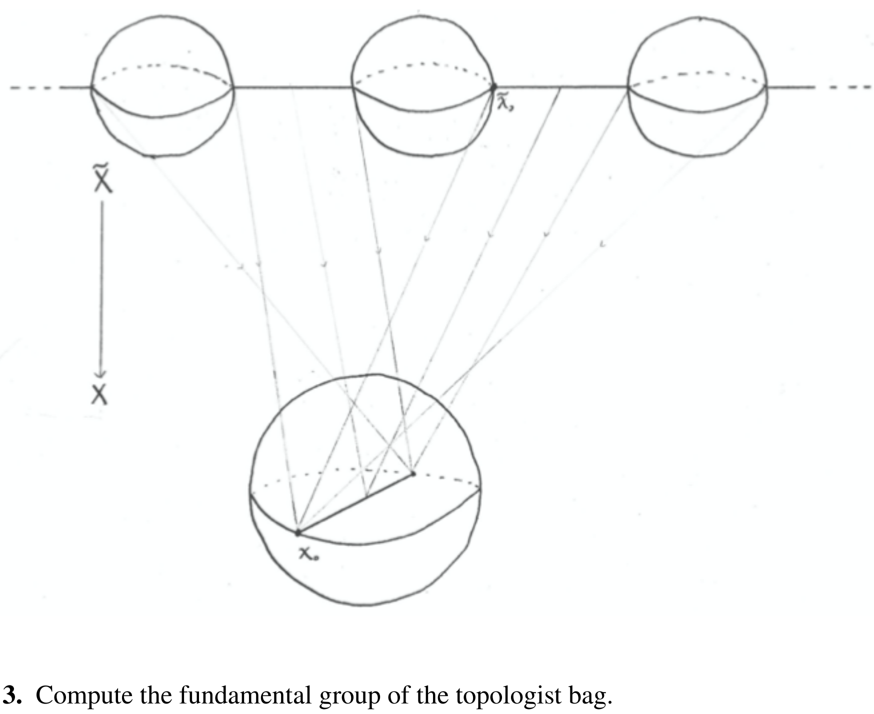


Proof. If we choose the second representation of the topologist bag in the picture above, it is evident that the map

$$\phi : \mathbb{R}^3 \setminus \{0\} \to \mathbb{R}^3 \setminus \{0\},$$

mapping $v \mapsto \frac{v}{|v|^2}$ switches the interior of the sphere with its esterior and transforms the inner edge into an external one. □

Exercise 2. Exhibit a simply connected covering of the topologist bag.



Exercise 3. Compute the fundamental group of the topologist bag.

Proof. The group of integers acts on the simply connected covering by translation. □

Remark 6. So, we are very powerful, we know how to compute many fundamental groups. But how the hell do we prove that S^n is simply connected? And in general, **if a space is simply connected, but is not contractible, how do we show it?** A strategy to attack this problem will be the main character of our last lesson in algebraic topology.

2. Exercises

Exercise 4. Exibit two connected (!) non homotopically equivalent spaces. You must also prove that they are not homotopically equivalent.

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Google the definition of *homotopically equivalent spaces*! Mind that there is a difference between homotopy equivalnece and *weak* homotopy equivalence.

Definition 7. The *pinched* sphere is a quotient of the sphere under the equivalence relation that identified the north pole to the south one $\mathsf{S}^2/\{N = S\}$.

Exercise 7. Compute the fundamental group of the pinched sphere.

Exercise 8. Exibit a simply connected covering of the pinched sphere.

Exercise 9. Show that the torus is not homeomorphic to the sphere. This example cannot be used for the red box.

Exercise 10 (⚠). Compute the fundamental group of 3×3 real matrices of rank 1.

- 🔴 the exercises in the red group are mandatory.
- 🟡 pick at least one exercise from each of the yellow groups.
- 🔵 pick at least two exercises from each of the blue groups.
- 🟤 nothing is mandatory in the brown box.
- 🟢 The riddle of the week. It's just there to let you think about it. It is not a mandatory exercise, nor it counts for your evaluation. Yet, it has a lot to teach.
- ☕ useful to deepen your understanding. Take your time to solve it. (May not be challenging at all.)
- ⚠ challenging.
- 📖 comes from **Elementary Topology Problem Textbook**, by *Viro, Ivanov, Netsvetaev and Kharlamov*.