TOPOLOGY

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ABSTRACT. This note summarizes the content of the fifth lesson of tutoring on the course Topology 2019. Also, attached at the end, there is an exercise sheet.

CONTENTS

1. CONNECTEDNESS

Hello, and welcome to the fifth lesson of this course in topology. The lesson of today is mainly based on **the book** (📕). In the last lesson I tried to give you an intuitive description of compactness. In the case of connectedness it is very hard to give a better description than the one that nature provided us with. A space is connected, if it is connected in the colloquial sense of the word. On the other hand, be careful, for humans it is quite easy to confuse connectedness with the very visual notion of *path connectedness*! I chose three topic where we can **apply connectedness** as a **technical tool**.

1.1. A collection of facts.

Before we start, let me recall a couple of relevant facts about connected spaces.

Fact 1 (Preservation of connectedness)**.** Let $\mathcal{X}$ and $\mathcal{Y}$ be topological spaces and let $f : \mathcal{X} \to \mathcal{Y}$ be a continuous function. If $\mathcal{X}$ is (path-)connected then the image $f(\mathcal{X})$ is (path-)connected.

Fact 2 (Characterization of clopens)**.** If a subset $A \subset \mathcal{X}$ is both open and closed, and $\mathcal{X}$ is connected, then A must be either $\emptyset$ or X itself.

1.2. Intermediate Value Theorem and its Generalizations.

The following theorem is usually included in Calculus. In fact, in a sense it is equivalent to connectedness of the segment.

Exercise 1 (Intermediate Value Theorem)**.** A continuous function $f : [a, b] \to \mathbb{R}$ takes every value between $f(a)$ and $f(b)$.

Proof. The statement is equivalent to show that

$$f([a, b]) = [f(a), f(b)].$$

This follows from Fact 1. Indeed, $f([a, b])$ must be connected, and contains $f(a)$ and $f(b)$. Now, assume that is strictly contained in $[f(a), f(b)]$, meaning that there exists a

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point $x \in [f(a), f(b)]$ that does not belong to $f([a, b])$. If so, the number of connected components of $f([a, b])$ is at least two (one containing $f(a)$ and the other containing $f(b)$). $f([a, b])$ must be connected! □

Remark 1. Many problems that can be solved by using the Intermediate Value Theorem can be found in Calculus textbooks. Here are few of them.

Exercise 2. Any polynomial of odd degree in one variable with real coefficients has at least one real root.

Proof. It's easy to see that any polynomial of odd degree must have at least a positive and a negative value. This means that $a, -b \in p(\mathbb{R})$. For the previous exercises, the whole interval $[-b, a]$ must be in the image of p, hence the thesis. □

Exercise 3. Let $\mathcal{X}$ be a connected space, $f : \mathcal{X} \to \mathbb{R}$ be a continuous function. Then $f(\mathcal{X})$ is an interval of $\mathbb{R}$.

Exercise 4. Let $J \subset \mathbb{R}$ be an interval of the real line and $f : J \to \mathbb{R}$ be a continuous function. Then $f(J)$ is also an interval of $\mathbb{R}$.[1]

1.3. **Applications to the Homeomorphism Problem.** Connectedness is a topological property, and the number of connected components is a topological invariant, this means that if two spaces are homeomorphic, then they should have the same number of connected components. Simple constructions assigning homeomorphic spaces to homeomorphic ones (e.g., deleting one or several points), allow us to use connectedness for proving that some connected spaces are not homeomorphic.

Exercise 5. $[0, 2]$ and $[0, 1] \cup [2, 3]$ are not homeomorphic.

Proof. Different number of connected components. □

Exercise 6. $[0, \infty)$ and $\mathbb{R}$, are not homeomorphic.

Proof. Assume there exists an homeomorphism ϕ. Then, also $[0, \infty) \setminus \{0\}$ should be homeomorphic to $\mathbb{R} \setminus \{\phi(0)\}$, by restricting ϕ. Yet, this is not possible, because $[0, \infty) \setminus \{0\}$ is connected c, while $\mathbb{R} \setminus \{\phi(0)\}$ has two connected components! (Draw a picture!) □

Definition 2 (The circle)**.** The circle S^1 is the following subset of the real plane.

$$S^1 := \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1.\}$$

Exercise 7. $[0, \infty)$ and S^1 are not homeomorphic.

Proof. Assume there exists an homeomorphism ϕ. Then, also $[0, \infty) \setminus \{5\}$ should be homeomorphic to $S^1 \setminus \{\phi(5)\}$, by restricting ϕ. Yet, this is not possible, because $[0, \infty) \setminus \{5\}$ has two connected components, while $S^1 \setminus \{\phi(5)\}$ has only one! (Draw a picture!) □

Exercise 8. A circle is not homeomorphic to a subspace of $\mathbb{R}$.

Exercise 9. That square and segment are not homeomorphic.

Proof. The square minus one point has one connected components, the segment has two. □

Exercise 10. $\mathbb{R}$ and $\mathbb{R}^n$ are not homeomorphic if $n > 1$.

Proof. $\mathbb{R}^n$ minus one point has one connected components, $\mathbb{R}$ has two. □

[1] In other words, continuous functions map intervals to intervals.

1.4. **Induction on Connectedness.** Here we see a typical proof technique in topology.

Definition 3. A map $f : \mathcal{X} \to \mathcal{Y}$ is locally constant if each point of X has a neighborhood U such that the restriction of f to U is constant.

Exercise 11. A (continuous) locally constant map on a connected set is constant.

Proof. The set of elements on which the map is constant is both open and closed (check it!). Thus the set must be either empty or coincide with the whole set. Since it is nonempty, it must coincide with the whole set. □

Exercise 12 (Induction on Connectedness)**.** Let E be a property of subsets of a topological space $\mathcal{X}$ such that the union of sets with nonempty pairwise intersections inherits this property from the sets involved. If $\mathcal{X}$ is connected and each point in $\mathcal{X}$ has a neighborhood with property E, then $\mathcal{X}$ also has property E.

Proof. Similar to the previous exercise. □

2. Exercises

The Book (📘). [12'4] G,H,J,K.

Exercise 13. Show that if two spaces are homeomorphic, then there exists a bijection between their sets of connected components.

Exercise 14. Any locally constant map is continuous.

Exercise 15. A connected manifold is path connected.

Exercise 16. Can you describe the connected components of a product $\mathcal{X} \times \mathcal{Y}$ in terms of the connected components of $\mathcal{X}$ and $\mathcal{Y}$? [a]

[a] For example, $\mathbb{R}_* = \mathbb{R} \setminus \{0\}$ has two connected components, how many connected components $(\mathbb{R}_*)^2$ does have?

Exercise 17. A space $\mathcal{X}$ is compact-connected if, given two points $a, b \in \mathcal{X}$ there exists a compact and connected subspace containing both of them. What is the relationship between path-connectedness, connectedness and compact-connectedness[a]? Give proofs of the implications, it is fine if you do not provide counterexamples to the converses.

The Book (📘). [14'9x] 14.27x.(1)

[a] Which one implies which one?

The riddle of the week (☕). Provide counterexamples to Ex. 17.

- 🔴 the exercises in the red group are mandatory.
- 🟡 pick at least one exercise from each of the yellow groups.
- 🔵 pick at least two exercises from each of the blue groups.
- 🟢 The riddle of the week. It's just there to let you think about it. It is not a mandatory exercise, nor it counts for your evaluation. Yet, it has a lot to teach.
- ☕ useful to deepen your understanding. Take your time to solve it. (May not be challenging at all.)
- ⚠ challenging.
- 📘 comes from **Elementary Topology Problem Textbook**, by *Viro, Ivanov, Netsvetaev and Kharlamov*.