TOPOLOGY

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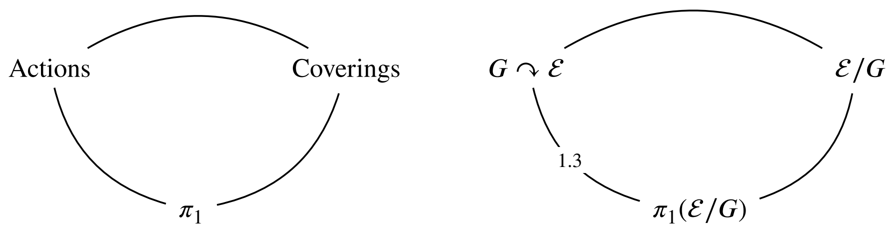
Abstract. This note summarizes the content of the 9th lesson of tutoring on the course Topology 2020. Also, attached at the end, there is an exercise sheet.

1. Actions, coverings, fundamental groups: an eternal golden braid

In the last lecture we concentrated on group actions and we used them to give nice (combinatorial) presentations of relevant spaces. Our favourite examples were the actions $\mathbb{Z}_2 \curvearrowright \mathsf{S}^2$ and $\mathbb{Z} \curvearrowright \mathbb{R}$. In this lecture we try to axiomatize the relevant properties of the quotient maps $\pi : \mathcal{X} \to \mathcal{X}/G$ when the action is nice enough. The goal of the lecture is to prove the following result:

Theorem (1.3)**.** Let $G \curvearrowright \mathcal{E}$ be a properly discontinuous action of a group on a (path connected) simply connected space $\mathcal{E}$. Then $G \cong \pi_1(\mathcal{X}/G, x)$.



1.1. Coverings.

Definition 1. A surjective continuous map $\pi : \mathcal{E} \twoheadrightarrow \mathcal{X}$ is a **covering** if for every point $x \in \mathcal{X}$ there exist a open neighbourhood V of x such that

$$\pi^{-1}(V) = \coprod_i V_i$$

and π is a homeomorphism when restricted to each V_i.

Remark 2. We shall list some terminology, and usual additional assumptions.

(1) V is called *trivializing open* or *local trivialization*.
(2) For every V_i there is a map $\pi_i^{-1} : V_i \to V$ that locally inverts π.
(3) The definition is actually interesting when $\mathcal{E}$ is connected, thus we will assume it for the whole lesson.
(4) Coverings are local homeomorphisms.

Remark 3 (Draw!)**.** In the following examples and for the whole lecture we highly recommend to draw pictures of every map and space involved.

Example 4 (The projective space)**.** Last week we gave a presentation of the projective space $\mathbb{RP}^n$ as a quotient of S^n under the action of the antipodal map. A picture will convince you that the quotient map $\mathsf{S}^n \to \mathbb{RP}^n$ is a covering.

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Example 5 (The circle)**.** The same can be said for the circle, when seen as a quotient of the line under the action of $\mathbb{Z}$. In that case it is enough to chose $B(x, \frac{1}{5})$ to trivialize the covering in each x.

1.1.1. *Lifts.* The rest of the section is devoted to study some relevant properties of coverings that will turn our to be useful later.

Definition 6. In the diagram below, let π be a covering and f be a continuous function. A **lift** $\tilde{f}$ of f is a continuous map such that $\tilde{f} \circ \pi = f$.

$$\begin{array}{ccc} & & \mathcal{E} \\ & \overset{\tilde{f}}{\nearrow} & \downarrow \pi \\ \mathcal{Y} & \xrightarrow{f} & \mathcal{X} \end{array}$$

Remark 7 (Rigidity of lifts)**.** When $\mathcal{Y}$ is connected, as soon as two lifts of the same map coincide, they coincide everywhere. We prove it under the additional hypothesis that $\mathcal{X}$ is Hausdorff. Call g and h the two lifts. We shall prove that the set of points on which they coincide Γ_{hg} is both open and closed, the thesis follows because $\mathcal{Y}$ is connected. Γ_{hg} is clearly closed because $\mathcal{X}$ is T_2. On the other hand it is also open, in fact we show that if x belongs to Γ_{hg}, then a whole neighbourhood V of x belongs to Γ_{hg}. Consider a local trivialization U of $f(x)$, then h and g will coincide with some $f \circ \pi_i^{-1}$, but the i must be the same and is controlled by the image of x. Thus h and g will coincide on $V = f^{-1}(U)$.

Proposition 1.1 (Lifting loops)**.** Let $\pi : \mathcal{E} \to \mathcal{X}$ be a covering and $\gamma : [0, 1] \to \mathcal{X}$ be a closed loop with starting point x, then for every point $e \in \pi^{-1}(x)$ there exist a (unique) lift $\tilde{\gamma}$ of γ such that $\tilde{\gamma}_e(0) = e$.

Sketcky proof. If $\gamma(I)$ is contained in a local trivialization V of π at x, then the lift clearly exist and is controlled by the position of e in the counterimage of V. The situation looks more complicated when $\gamma(I)$ is not contained in a local trivialization. In that case, we find a finite family $[a_i, b_i]$ of subintervals such that $f[a_i, b_i]$ is contained in a local trivialization[1], and we lift each one of these arcs. When lifting, be careful, because $b_i = a_{i+1}$! □

1.1.2. *Monodromy: bridging coverings and fundamental groups.* Let $\pi : \mathcal{E} \to \mathcal{X}$ be a covering. In this very brief subsection we set a connection between the fundamental group of $\mathcal{X}$ and the fibers of the covering. This is the conceptual brick of covering theory.

Proposition 1.2. Let $\pi : \mathcal{E} \to \mathcal{X}$ be a covering. Then $\pi_1(X, x)$ acts (on the right) on $\pi^{-1}(x)$. This is called **monodromy action**.

Proof. We just define the action, we have no time to provide a detailed proof.

$$e \cdot [\gamma] := \tilde{\gamma}_e(1).$$

□

1.2. **Properly discontinuous actions.** In this section we prove a very important result, we show that when G acts nicely on a simply connected space $\mathcal{E}$, then it completely identifies the fundamental group of the quotient. This is a very good tool to compute the fundamental group of those spaces for which we have a combinatorial presentation.

Definition 8. An action $\chi : G \curvearrowright \mathcal{E}$ is **properly discontinuous** if for every) $e \in \mathcal{E}$ and every (non trivial) $g \in G$ there exist an open set U such that $g(U) \cap U = \emptyset$.

[1] It is not completely trivial to prove that we can find this family, most of the proof depends on the fact that I is a compact metric space. Try yourself to fill the details! If you do not manage to, look online for the *Lebesgue number*.

Theorem 1.3 (Main theorem)**.** Let $G \curvearrowright \mathcal{E}$ be a properly discontinuous action of a group on a (path connected) simply connected space $\mathcal{E}$. Then $G \cong \pi_1(\mathcal{E}/G, x)$.

Proof. We build a group homomorphism $\chi : G \to \pi_1(\mathcal{E}/G, x)$. Consider your favourite point of $e \in \pi^{-1}(x)$ and an element $g \in G$, we want to define $\chi(g)$. Since $\mathcal{E}$ is path connected, there is a path γ_g connecting e and $g \cdot e$. $\pi(\gamma)$ is a loop in $\pi_1(\mathcal{E}/G, x)$, thus we could define

$$\chi(g) := \pi \circ \gamma_g.$$

Def This definition is apparently problematic, because in principle we could choose δ_g (another path) to witness the topological proximity of e and $g \cdot e$. Thanks to god, or whatever prodigy you like the most, $\gamma_g \star \delta_g^{-1}$ is a loop in $\pi_1(\mathcal{E}, e)$ and thus is null-homotopic, because $\mathcal{E}$ is simply connected. That's the same of saying that the two paths are homotopic via a homotopy H. Since H is a homotoby between γ_g and δ_g, $\pi \circ H$ will be a homotopy between $\pi \circ \gamma_g$ and $\pi \circ \delta_g$, thus $\chi(g)$ is a well defined element of $\pi_1(\mathcal{X}/G, x)$. It is very easy to check that χ is a group homomorphism.

Inj We show that $\mathrm{Ker}(\chi)$ is trivial. Let g be an element such that $\chi(g) = \mathrm{id}$. Then $\chi(g)$ is a lift of the constant path in $\mathcal{E}/G$. By the uniqueness of the lift, $g \cdot e = e$, and since the action is properly discontinuous g is globally the identity, that is the thesis.

Sur This is probably the easiest part of the proof. Let γ be a path in $\pi_1(\mathcal{E}/G, x)$ and consider a lift $\tilde{\gamma}_e$. Since, by definition of $\mathcal{E}/G$, there is a $g \in G$ such that $\tilde{\gamma}(1) = g \cdot e$, we get via simple computations that $\chi(g) = \gamma$.

□

Corollary 1.4. The fundamental group of the circle is $\mathbb{Z}$.

Remark 9. Today we went very briefly through the theory of **covering spaces**, choosing the shortest path to the main theorem, but we did not exhaust the theory, which is quite rich and definitively beautiful, we recommend further investigations to the interested student.

2. Exercises

Exercise 1. Prove that the quotient map $\mathsf{S}^n \twoheadrightarrow \mathbb{RP}^n$ is a covering.

Exercise 2. Prove that if $\pi : \mathcal{E} \to \mathcal{X}$ and $p : \mathcal{F} \to \mathcal{Y}$ are coverings, then $\pi \times p : \mathcal{E} \times \mathcal{F} \to \mathcal{X} \times \mathcal{Y}$ is a covering too.

Exercise 3. Prove that $\pi_1(\mathcal{X} \times \mathcal{Y}) \cong \pi_1(\mathcal{X}) \times \pi_1(\mathcal{Y})$.

Exercise 4. Find a simply connected[a] covering of $\mathbb{R}^2 \setminus 0$.[b]

Exercise 5. Find a simply connected covering of the torus.[c]

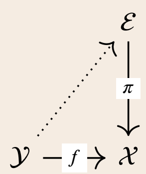
[a] A space is *simply connected* if its fundamental group is trivial. A covering is simply connected if the domain is so.
[b] Hint: you may want to use Ex. 2 and 3.
[c] Hint: you may want to use Ex. 2 and 3. Recall that the torus is $\mathsf{S}^1 \times \mathsf{S}^1$.

Exercise 6. Prove that a non surjective loop on the sphere is null-homotopic[a].

Exercise 7. Prove that $\mathbb{R}^3$ minus a line deforms[b] over the sphere minus two points.

[a] To be more precise, call $\gamma : [0, 1] \to \mathsf{S}^2$ such a loop and prove that γ is homotopically equivalent to the constant loop.
[b] You do not know what is a *deformation retract*? Google it.

Exercise 8 (☕). In the diagram below, assume that π is a covering, $\mathcal{X}$ and $\mathcal{Y}$ path connected.



Call f_* and π_* the induced maps between the fundamental groups. Show that if f has a lifting, then $\mathrm{Im}(f_*) \subset \mathrm{Im}(\pi_*)$.

Exercise 9 (☕). What about the viceversa?

- the exercises in the red group are mandatory.
- pick at least one exercise from each of the yellow groups.
- pick at least two exercises from each of the blue groups.
- nothing is mandatory in the brown box.
- The riddle of the week. It's just there to let you think about it. It is not a mandatory exercise, nor it counts for your evaluation. Yet, it has a lot to teach.
- ☕ useful to deepen your understanding. Take your time to solve it. (May not be challenging at all.)
- ⚠ challenging.
- 📖 comes from **Elementary Topology Problem Textbook**, by *Viro, Ivanov, Netsvetaev and Kharlamov*.