Elementary Topology Problem Textbook

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Dedicated to the memory of Vladimir Abramovich Rokhlin (1919–1984)
– our teacher

Contents

Introduction

The subject of the book: Elementary Topology

Elementary means close to elements, basics. It is impossible to determine precisely, once and for all, which topology is elementary and which is not. The elementary part of a subject is the part with which an expert starts to teach a novice.

We suppose that our student is ready to study topology. So, we do not try to win her or his attention and benevolence by hasty and obscure stories about mysterious and attractive things such as the Klein bottle,[1] though the Klein bottle will appear in its turn. However, we start with what a topological space is, that is, we start with general topology.

General topology became a part of the general mathematical language a long time ago. It teaches one to speak clearly and precisely about things related to the idea of continuity. It is not only needed to explain what, finally, the Klein bottle is, but it is also a way to introduce geometrical images into any area of mathematics, no matter how far from geometry the area may be at first glance.

As an active research area, general topology is practically completed. A permanent usage in the capacity of a general mathematical language has polished its system of definitions and theorems. Indeed, nowadays, the study of general topology resembles a study of a language rather than a study of mathematics: one has to learn many new words, while the proofs of the majority of the theorems are extremely simple. However, the quantity of

[1] A person who is looking for such elementary topology will easily find it in numerous books with beautiful pictures on visual topology.

the theorems is huge. This comes as no surprise because they play the role of rules that regulate usage of words.

The book consists of two parts. General topology is the subject of part one. The second part is an introduction to algebraic topology via its most classical and elementary segment, which emerges from the notions of fundamental group and covering space.

In our opinion, elementary topology also includes basic topology of manifolds, i.e., spaces that look locally as the Euclidean space. One- and two-dimensional manifolds, i.e., curves and surfaces are especially elementary. However, a book should not be too thick, and so we had to stop.

Chapter 5, which is the last chapter of the first part, keeps somewhat aloof. It is devoted to topological groups. The material is intimately related to a number of different areas of Mathematics. Although topological groups play a profound role in those areas, it is not that important in the initial study of general topology. Therefore, mastering this material may be postponed until it appears in a substantial way in other mathematical courses (which will concern the Lie groups, functional analysis, etc.). The main reason why we included this material is that it provides a great variety of examples and exercises.

Organization of the text

Even a cursory overview detects unusual features in the organization of this book. We dared to come up with several innovations and hope that the reader will quickly get used to them and even find them useful.

We know that the needs and interests of our readers vary, and realize that it is very difficult to make a book interesting and useful for *each* reader. To solve this problem, we formatted the text in such a way that the reader could easily determine what (s)he can expect from each piece of the text. We hope that this will allow the reader to organize studying the material of the book in accordance with his or her tastes and abilities. To achieve this goal, we use several tricks.

First of all, we distinguished the basic, so to speak, lecture line. This is the material which we consider basic. It constitutes a minor part of the text.

The basic material is often interrupted by specific examples, illustrative and training problems, and discussion of the notions that are related to these examples and problems, but are not used in what follows. Some of the notions play a fundamental role in other areas of mathematics, but here they are of minor importance.

In a word, the basic line is interrupted by *variations* wherever possible. The variations are clearly separated from the *basic theme* by graphical means.

The second feature distinguishing the present book from the majority of other textbooks is that proofs are separated from formulations. Proofs were removed entirely from the web version. This makes the book look like a pure problem book. It would be easy to make the book looking like hundreds of other mathematical textbooks. For this purpose, it suffices to move all variations to the ends of their sections so that they would look like exercises to the basic text, and put the proofs of theorems immediately after their formulations.

For whom is this book?

A reader who has safely reached the university level in her/his education may bravely approach this book. Super brave daredevils may try it even earlier. However, we cannot say that no preliminary knowledge is required. We suppose that the reader is familiar with real numbers, and, surely, with natural, integer, and rational numbers too. A knowledge of complex numbers would also be useful, although one can manage without them in the first part of the book.

We assume that the reader is acquainted with naive set theory, but admit that this acquaintance may be superficial. For this reason, we make special set-theoretical digressions where the knowledge of set theory is particularly desirable.

We do not seriously rely on calculus, but because the majority of our readers are already familiar with it, at least slightly, we do not hesitate to resort to using notations and notions from calculus.

In the second part, experience in group theory will be useful, although we give all necessary information about groups.

One of the most valuable acquisitions that the reader can make by mastering the present book is new elements of mathematical culture and an ability to understand and appreciate an abstract axiomatic theory. The higher the degree in which the reader already possesses this ability, the easier it will be for her or him to master the material of the book.

If you want to study topology on your own, do try to work with the book. It may turn out to be precisely what you need. However, you should attentively reread the rest of the Introduction again in order to understand how the material is organized and how you can use it.

The basic theme

The core of the book is made up of the material of the topology course for students majoring in Mathematics at the Saint Petersburg (Leningrad) State University. The core material makes up a relatively small part of the book and involves nearly no complicated arguments.

The reader should not think that by selecting the basic theme the authors just try to impose their tastes on her or him. We do not hesitate to do this occasionally, but here our primary goal is to organize study of the subject.

The basic theme forms a complete entity. The reader who has mastered the basic theme has mastered the subject. Whether the reader had looked in the variations or not is her or his business. However, the variations have been included in order to help the reader with mastering the basic material. They are not exiled to the final pages of sections in order to have them at hand precisely when they are most needed. By the way, the variations can tell you about many interesting things. However, following the variations too literally and carefully may take far too long.

We believe that the material presented in the basic theme is the minimal amount of topology that must be mastered by every student who has decided to become a professional mathematician.

Certainly, a student whose interests will be related to topology and other geometrical disciplines will have to learn far more than the basic theme includes. In this case the material can serve as a good starting point.

For a student who is not going to become a professional mathematician, even a selective acquaintance with the basic theme might be useful. It may be useful for preparation for an exam or just for catching a glimpse and a feeling of abstract mathematics, with its emphasized value of definitions and precise formulations.

Where are the proofs?

The book is tailored for a reader who is determined to work actively.

The proofs of theorems are separated from their formulations and placed at the end of the current chapter. They are not included at all in the web version of the book.

We believe that the first reaction to the formulation of any assertion (coming immediately after the feeling that the formulation has been understood) must be an attempt to prove the assertion—or to disprove it, if you do not manage to prove it. An attempt to disprove an assertion may be useful both for achieving a better understanding of the formulation and for looking for a proof.

By keeping the proofs away from the formulations, we want to encourage the reader to think through each formulation, and, on the other hand, to make the book inconvenient for careless skimming. However, a reader who prefers a more traditional style and, for some reason, does not wish to work too actively can either find the proofs at the end of the chapter, or skip them all together. (Certainly, in the latter case there is some danger of misunderstanding.)

This style can also please an expert who needs a handbook and prefers formulations not overshadowed by proofs. Most of the proofs are simple and easy to discover.

Structure of the book

Basic structural units of the book are sections. They are divided into numbered and titled subsections. Each subsection is devoted to a single topic and consists of definitions, comments, theorems, exercises, problems, and riddles.

By a *riddle* we mean a problem whose solution (and often also the meaning) should be guessed rather than calculated or deduced from the formulation.

Theorems, exercises, problems, and riddles belonging to the basic material are numbered by pairs consisting of the number of the current section and a letter, separated by a dot.

2.B. Riddle. Taking into account the number of the riddle, determine in which section it must be contained. By the way, is this really a riddle?

The letters are assigned in alphabetical order. They number the assertions inside a section.

A difficult problem (or theorem) is often followed by a sequence of assertions that are lemmas to the problem. Such a chain often ends with a problem in which we suggest the reader, armed with the lemmas just proven, return to the initial problem (respectively, theorem).

Variations

The basic material is surrounded by numerous training problems and additional definitions, theorems, and assertions. In spite of their relation to the basic material, they usually are left outside of the standard lecture course.

Such additional material is easy to recognize in the book by the smaller print and wide margins, as shown here. Exercises, problems, and riddles that are not included in the basic material, but are closely related to it, are numbered by pairs consisting of the number of a section and the number of the assertion in the limits of the section.

2.5. Find a problem with the same number *2.5* in the main body of the book.

All solutions to problems are located at the end of the book published by AMS. They are not included into the web version.

As is common, the problems that have seemed to be most difficult to the authors are marked by an asterisk. They are included with different purposes: to outline relations to other areas of mathematics, to indicate possible directions of development of the subject, or just to please an ambitious reader.

Additional themes

We decided to make accessible for interested students certain theoretical topics complementing the basic material. It would be natural to include them into lecture courses designed for senior (or graduate) students. However, this does not usually happen, because the topics do not fit well into traditional graduate courses. Furthermore, studying them seems to be more natural during the very first contacts with topology.

In the book, such topics are separated into individual subsections, whose numbers contain the symbol *x*, which means *extra*. (Sometimes, a whole section is marked in this way, and, in one case, even a whole chapter.)

Certainly, regarding this material as additional is a matter of taste and viewpoint. Qualifying a topic as additional, we follow our own ideas about what must be contained in the initial study of topology. We realize that some (if not most) of our colleagues may disagree with our choice, but we hope that our decorations will not hinder them from using the book.

Advices to the reader

You can use the present book when preparing for an exam in topology (especially so if the exam consists in solving problems). However, if you attend lectures in topology, then it is reasonable to read the book before the lectures, and try to prove the assertions in it on your own before the lecturer will prove them.

The reader who can prove assertions of the basic theme on his or her own needn't solve *all* of the problems suggested in the variations, and can resort to a brief acquaintance with their formulations and solve only the most difficult of them. On the other hand, the more difficult it is for you to prove assertions of the basic theme, the more attention you should pay to illustrative problems, and the less attention should be paid to problems with an asterisk.

Many of our illustrative problems are easy to come up with. Moreover, when seriously studying a subject, one should permanently cook up questions of this kind.

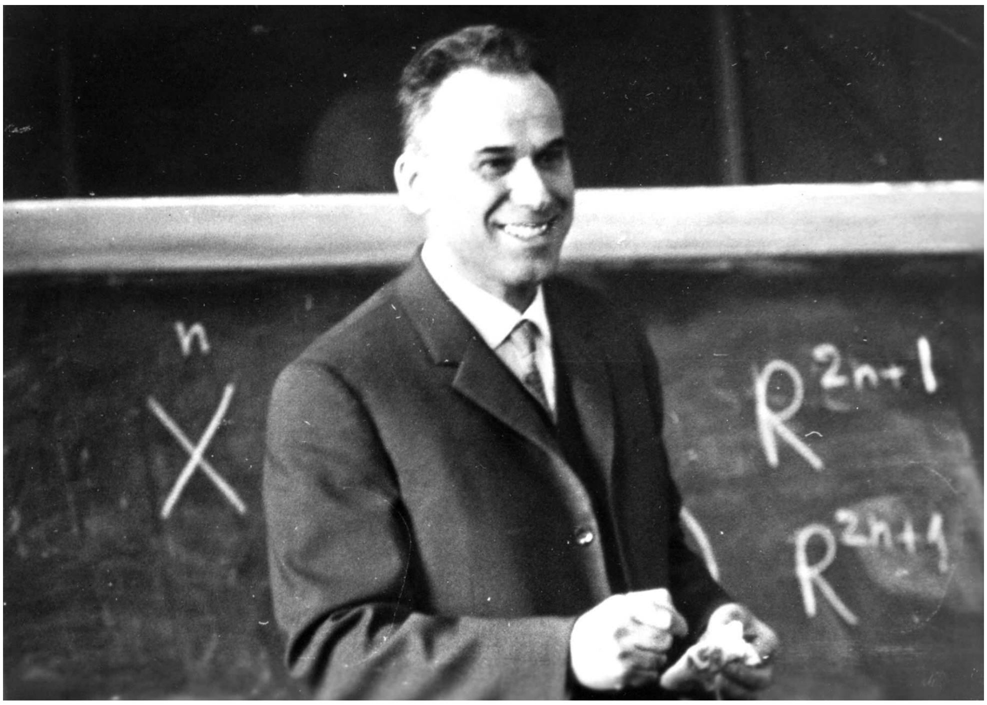
On the other hand, some problems presented in the book are not easy to come up with at all. We have widely used all kinds of sources, including both literature and teachers' folklore.

Notations

We did our best to avoid notations which are not commonly accepted. The only exception is the use of a few symbols which are very convenient and almost self-explanatory. Namely, within proofs symbols $\boxed{\Longrightarrow}$ and $\boxed{\Longleftarrow}$ should be understood as (sub)titles. Each of them means that we start proving the corresponding implication. Similarly, symbols $\boxed{\subset}$ and $\boxed{\supset}$ indicate the beginning of proofs of the corresponding inclusions.

How this book was created

In the basic theme, we follow the course of lectures composed by Vladimir Abramovich Rokhlin at the Faculty of Mathematics and Mechanics of the Leningrad State University in the 1960s. It seems appropriate to sketch the circumstances of creating the course, although we started to write this book only after Vladimir Abramovich's death (1984).



Vladimir Abramovich Rokhlin gives a lecture, 1960s.

In the 1960s, mathematics was one of the most attractive areas of science for young people in the Soviet Union, being second maybe only to physics among the natural sciences. Every year more than a hundred students were enrolled in the mathematical subdivision of the Faculty.

Several dozen of them were alumnae and alumni of mathematical schools. The system and contents of the lecture courses at the Faculty were seriously updated.

Until Rokhlin developed his course, topology was taught in the Faculty only in the framework of special courses. Rokhlin succeeded in including a one-semester course on topology into the system of general mandatory courses. The course consisted of three chapters devoted to general topology, fundamental group and coverings, and manifolds, respectively. The contents of the first two chapters differed only slightly from the basic material of the book. The last chapter started with a general definition of a topological manifold, included a topological classification of one-dimensional manifolds, and ended either with a topological classification of triangulated two-dimensional manifolds or with elements of differential topology, up to embedding a smooth manifold in the Euclidean space.

Three of the four authors belong to the first generation of students who attended Rokhlin's lecture course. This was a one-semester course, three hours a week in the first semester of the second year. At most two two-hour lessons during the whole semester were devoted to solving problems. It was not Rokhlin, but his graduate students who conducted these lessons. For instance, in 1966–68 they were conducted by Misha Gromov—an outstanding geometer, currently a professor of the Paris Institute des Hautes Etudies Scientifiques and the New York Courant Institute. Rokhlin regarded the course as a theoretical one and did not wish to spend lecture time solving problems. Indeed, in the framework of the course one did not have to teach students how to solve series of routine problems, like problems in techniques of differentiation and integration, that are traditional for calculus.

Despite the fact that we built our book by starting from Rokhlin's lectures, the book will give you no idea about Rokhlin's style. The lectures were brilliant. Rokhlin wrote very little on the blackboard. Nevertheless, it was very easy to take notes. He spoke without haste, with maximally simple and ideally correct sentences.

For the last time, Rokhlin gave his mandatory topology course in 1973. In August of 1974, because of his serious illness, the administration of the Faculty had to look for a person who would substitute for Rokhlin as a lecturer. The problem was complicated by the fact that the results of the exams in the preceding year were terrible. In 1973, the time allotted for the course was increased up to four hours a week, while the number of students had grown, and, respectively, the level of their training had decreased. As a result, the grades for exams "crashed down".

It was decided that the whole class, which consisted of about 175 students, should be split into two classes. Professor Viktor Zalgaller was appointed to give lectures to the students who were going to specialize in applied mathematics, while Assistant Professor Oleg Viro would give the lectures to student-mathematicians. Zalgaller suggested introducing exercise lessons—one hour a week. As a result, the time allotted for the lectures decreased, and de facto the volume of the material also reduced along with the time.

It remained to understand *what* to do in the exercise lessons. One had to develop a system of problems and exercises that would give an opportunity to revisit the definitions given in the lectures, and would allow one to develop skills in proving easy theorems from general topology in the framework of a simple axiomatic theory.

Problems in the first part of the book are a result of our efforts in this direction. Gradually, exercise lessons and problems were becoming more and more useful as long as we had to teach students with a lower level of preliminary training. In 1988, the Publishing House of the Leningrad State University published the problems in a small book, *Problems in Topology*.

Students found the book useful. One of them, Alekseĭ Solov'ev, even translated it into English on his own initiative when he became a graduate student at the University of California. The translation initiated a new stage of work on the book. We started developing the Russian and English versions in parallel and practically covered the entire material of Rokhlin's course. In 2000, the Publishing House of the Saint Petersburg State University published the second Russian edition of the book, which already included a chapter on the fundamental group and coverings.

The English version was used by Oleg Viro for his lecture course in the USA (University of California) and Sweden (Uppsala University). The Russian version was used by Slava Kharlamov for his lecture courses in France (Strasbourg University). The lectures have been given for quite different audiences: both for undergraduate and graduate students. Furthermore, few professors (some of whom the authors have not known personally) have asked the authors' permission to use the English version in their lectures, both in the countries mentioned above and in other ones. New demands upon the text have arisen. For instance, we were asked to include solutions to problems and proofs of theorems in the book, in order to make it meet the Western standards and transform it from a problem book into a self-sufficient textbook. After some hesitation, we fulfilled those requests, the more so that they were upheld by the Publishing House of the American Mathematical Society.

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Each of us has been lucky to be a student of Vladimir Abramovich Rokhlin, to whose memory we dedicate this book.

The authors, from left to right:

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Part 1

General Topology

Our goal in this part of the book is to teach the basics of the mathematical language. More specifically, one of its most important components: the language of set-theoretic topology, which treats the basic notions related to continuity. The term *general topology* means: this is the topology that is needed and used by most mathematicians. The permanent usage in the capacity of a common mathematical language has polished its system of definitions and theorems. Nowadays, studying general topology really more resembles studying a language rather than mathematics: one needs to learn a lot of new words, while proofs of most theorems are quite simple. On the other hand, the theorems are numerous because they play the role of rules regulating usage of words.

We have to warn students for whom this is one of their first mathematical subjects. Do not hurry to fall in love with it. Do not let an imprinting happen. This field may seem to be charming, but it is not very active nowadays. Other mathematical subjects are also nice and can give exciting opportunities for research. Check them out!

Chapter I

Structures and Spaces

1. Set-Theoretic Digression: Sets

We begin with a digression, which, however, we would like to consider unnecessary. Its subject is the first basic notions of the naive set theory. This is a part of the common mathematical language, too, but an even more profound part than general topology. We would not be able to say anything about topology without this part (look through the next section to see that this is not an exaggeration). Naturally, it may be expected that the naive set theory becomes familiar to a student when she or he studies Calculus or Algebra, two subjects of study that usually precede topology. If this is true in your case, then, please, just glance through this section and pass to the next one.

⌈1′1⌋ Sets and Elements

In an intellectual activity, one of the most profound actions is gathering objects in groups. The gathering is performed in mind and is not accompanied with any action in the physical world. As soon as the group has been created and assigned a name, it can be a subject of thoughts and arguments and, in particular, can be included into other groups. Mathematics has an elaborate system of notions, which organizes and regulates creating those groups and manipulating them. The system is called the *naive set theory*, which, however, is a slightly misleading name because this is rather a language than a theory.

The first words in this language are *set* and *element*. By a set we understand an arbitrary collection of various objects. An object included in the collection is an *element* of the set. A set *consists* of its elements. It is also *formed* by them. In order to diversify the wording, the word *set* is replaced by the word *collection*. Sometimes other words, such as *class*, *family*, and *group*, are used in the same sense, but this is not quite safe because each of these words is associated in modern mathematics with a more special meaning, and hence should be used instead of the word *set* with caution.

If x is an element of a set A, then we write $x \in A$ and say that x *belongs to* A and A *contains* x. The sign $\in$ is a variant of the Greek letter epsilon, which corresponds to the first letter of the Latin word *element*. To make the notation more flexible, the formula $x \in A$ is also allowed to be written in the form $A \ni x$. So, the origin of the notation is sort of ignored, but a more meaningful similarity to the inequality symbols $<$ and $>$ is emphasized. To state that x is not an element of A, we write $x \notin A$ or $A \not\ni x$.

⌈1′2⌋ Equality of Sets

A set is determined by its elements. The set is nothing but a collection of its elements. This manifests most sharply in the following principle: *two sets are considered equal if and only if they have the same elements*. In this sense, the word *set* has slightly disparaging meaning. When something is called a set, this shows, maybe unintentionally, a lack of interest to whatever organization of the elements of this set.

For example, when we say that a line is a set of points, we assume that two lines coincide if and only if they consist of the same points. On the other hand, we commit ourselves to consider all relations between points on a line (e.g., the distance between points, the order of points on the line, etc.) separately from the notion of a line.

We may think of sets as boxes that can be built effortlessly around elements, just to distinguish them from the rest of the world. The cost of this lightness is that such a box is not more than the collection of elements placed inside. It is a little more than just a name: it is a declaration of our wish to think about this collection of things as an entity and not to go into details about the nature of its member-elements. Elements, in turn, may also be sets, but as long as we consider them elements, they play the role of atoms, with their own original nature ignored.

In modern mathematics, the words *set* and *element* are very common and appear in most texts. They are even overused. There are instances when it is not appropriate to use them. For example, it is not good to use the word *element* as a replacement for other, more meaningful words. When you call something an *element*, then the *set* whose element is this one should be clear. The word *element* makes sense only in combination with the word *set*, unless we deal with a nonmathematical term (like *chemical element*), or a rare old-fashioned exception from the common mathematical terminology (sometimes the expression under the sign of integral is called an *infinitesimal element*; lines, planes, and other geometric images are also called *elements* in old texts). Euclid's famous book on geometry is called *Elements*, too.

⌈1′3⌋ The Empty Set

Thus, an element may not be without a set. However, a set may have no elements. Actually, there is such a set. This set is unique because a set is completely determined by its elements. It is the ***empty set*** denoted[1] by $\varnothing$.

⌈1′4⌋ Basic Sets of Numbers

In addition to $\varnothing$, there are some other sets so important that they have their own special names and designations. The set of all positive integers, i.e., 1, 2, 3, 4, 5, ..., etc., is denoted by $\mathbb{N}$. The set of all integers, both positive, and negative, and zero, is denoted by $\mathbb{Z}$. The set of all rational numbers (add to the integers the numbers that are presented by fractions, like 2/3 and $\frac{-7}{5}$) is denoted by $\mathbb{Q}$. The set of all real numbers (obtained by adjoining to rational numbers the numbers like $\sqrt{2}$ and $\pi = 3.14\dots$) is denoted by $\mathbb{R}$. The set of complex numbers is denoted by $\mathbb{C}$.

⌈1′5⌋ Describing a Set by Listing Its Elements

A set presented by a list a, b, ..., x of its elements is denoted by the symbol $\{a, b, \dots, x\}$. In other words, the list of objects enclosed in curly brackets denotes the set whose elements are listed. For example, $\{1, 2, 123\}$ denotes the set consisting of the numbers 1, 2, and 123. The symbol $\{a, x, A\}$ denotes the set consisting of three elements: a, x, and A, whatever objects these three letters denote.

1.1. What is $\{\varnothing\}$? How many elements does it contain?

1.2. Which of the following formulas are correct:

1) $\varnothing \in \{\varnothing, \{\varnothing\}\}$; 2) $\{\varnothing\} \in \{\{\varnothing\}\}$; 3) $\varnothing \in \{\{\varnothing\}\}$?

A set consisting of a single element is a ***singleton***. This is any set which is presented as $\{a\}$ for some a.

1.3. Is $\{\{\varnothing\}\}$ a singleton?

[1] Other designations, like Λ, are also in use, but $\varnothing$ has become a common one.

Notice that the sets $\{1, 2, 3\}$ and $\{3, 2, 1, 2\}$ are equal since they have the same elements. At first glance, lists with repetitions of elements are never needed. There even arises a temptation to prohibit usage of lists with repetitions in such notation. However this would not be wise. In fact, quite often one cannot say *a priori* whether there are repetitions or not. For example, the elements in the list may depend on a parameter, and under certain values of the parameter some entries of the list coincide, while for other values they don't.

1.4. How many elements do the following sets contain?

1) $\{1, 2, 1\}$;	2) $\{1, 2, \{1, 2\}\}$;	3) $\{\{2\}\}$;
4) $\{\{1\}, 1\}$;	5) $\{1, \varnothing\}$;	6) $\{\{\varnothing\}, \varnothing\}$;
7) $\{\{\varnothing\}, \{\varnothing\}\}$;	8) $\{x, 3x - 1\}$ for $x \in \mathbb{R}$.	

⌈1′6⌋ Subsets

If A and B are sets and every element of A also belongs to B, then we say that A is a ***subset*** of B, or B ***includes*** A, and write $A \subset B$ or $B \supset A$.

The inclusion signs $\subset$ and $\supset$ resemble the inequality signs $<$ and $>$ for a good reason: in the world of sets, the inclusion signs are obvious counterparts for the signs of inequalities.

1.A. Let a set A have a elements, and let a set B have b elements. Prove that if $A \subset B$, then $a \leq b$.

⌈1′7⌋ Properties of Inclusion

1.B Reflexivity of Inclusion. Any set includes itself: $A \subset A$ holds true for any A.

Thus, the inclusion signs are not completely true counterparts of the inequality signs $<$ and $>$. They are closer to $\leq$ and $\geq$. Notice that no number a satisfies the inequality $a < a$.

1.C The Empty Set Is Everywhere. The inclusion $\varnothing \subset A$ holds true for any set A. In other words, the empty set is present in each set as a subset.

Thus, each set A has two obvious subsets: the empty set $\varnothing$ and A itself. A subset of A different from $\varnothing$ and A is a ***proper*** subset of A. This word is used when we do not want to consider the obvious subsets (which are ***improper***).

1.D Transitivity of Inclusion. If A, B, and C are sets, $A \subset B$, and $B \subset C$, then $A \subset C$.

⌈1′8⌋ To Prove Equality of Sets, Prove Two Inclusions

Working with sets, we need from time to time to prove that two sets, say A and B, which may have emerged in quite different ways, are equal. The most common way to do this is provided by the following theorem.

1.E Criterion of Equality for Sets.
$A = B$ *if and only if* $A \subset B$ *and* $B \subset A$.

⌈1′9⌋ Inclusion Versus Belonging

1.F. $x \in A$ *if and only if* $\{x\} \subset A$.

Despite this obvious relation between the notions of belonging $\in$ and inclusion $\subset$ and similarity of the symbols $\in$ and $\subset$, the concepts are quite different. Indeed, $A \in B$ means that A is an element in B (i.e., one of the indivisible pieces constituting B), while $A \subset B$ means that A is made of some of the elements of B.

In particular, we have $A \subset A$, while $A \notin A$ for any reasonable A. Thus, *belonging is not reflexive*. One more difference: *belonging is not transitive*, while inclusion is.

1.G Non-Reflexivity of Belonging. Construct a set A such that $A \notin A$. Cf. *1.B*.

1.H Non-Transitivity of Belonging. Construct three sets A, B, and C such that $A \in B$ and $B \in C$, but $A \notin C$. Cf. *1.D*.

⌈1′10⌋ Defining a Set by a Condition (Set-Builder Notation)

As we know (see Section 1′5), a set can be described by presenting a list of its elements. This simplest way may be not available or, at least, may not be the easiest one. For example, it is easy to say: "the set of all solutions of the following equation" and write down the equation. This is a reasonable description of the set. At least, it is unambiguous. Having accepted it, we may start speaking on the set, studying its properties, and eventually may be lucky to solve the equation and obtain the list of its solutions. (Though the latter task may be difficult, this should not prevent us from discussing the set.)

Thus, we see another way for a description of a set: to formulate properties that distinguish the elements of the set among elements of some wider and already known set. Here is the corresponding notation: the subset of a set A consisting of the elements x that satisfy a condition $P(x)$ is denoted by $\{x \in A \mid P(x)\}$.

1.5. Present the following sets by lists of their elements (i.e., in the form $\{a, b, \dots\}$)
(a) $\{x \in \mathbb{N} \mid x < 5\}$, (b) $\{x \in \mathbb{N} \mid x < 0\}$, (c) $\{x \in \mathbb{Z} \mid x < 0\}$.

⌈1′11⌋ Intersection and Union

The *intersection* of sets A and B is the set consisting of their common elements, i.e., elements belonging both to A and B. It is denoted by $A \cap B$ and is described by the formula

$$A \cap B = \{x \mid x \in A \text{ and } x \in B\}.$$

Two sets A and B are *disjoint* if their intersection is empty, i.e., $A \cap B = \varnothing$. In other words, they have no common elements.

The *union* of two sets A and B is the set consisting of all elements that belong to at least one of the two sets. The union of A and B is denoted by $A \cup B$. It is described by the formula

$$A \cup B = \{x \mid x \in A \text{ or } x \in B\}.$$

Here the conjunction *or* should be understood in the inclusive way: the statement "$x \in A$ or $x \in B$" means that x belongs to *at least one* of the sets A and B, and, maybe, to both of them.[2]

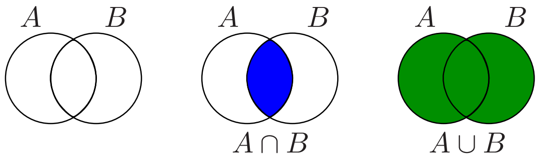


Figure 1. The sets A and B, their intersection $A \cap B$, and their union $A \cup B$.

1.I Commutativity of $\cap$ and $\cup$. For any two sets A and B, we have

$$A \cap B = B \cap A \quad \text{and} \quad A \cup B = B \cup A.$$

In the above figure, the first equality of Theorem *1.L* is illustrated by sketches. Such sketches are called *Venn diagrams* or *Euler circles*. They are quite useful, and we strongly recommend trying to draw them for each formula involving sets. (At least, for formulas with at most three sets, since in this case they can serve as proofs! (Guess why?)).

1.6. Prove that for any set A we have

$$A \cap A = A, \quad A \cup A = A, \quad A \cup \varnothing = A, \quad \text{and} \quad A \cap \varnothing = \varnothing.$$

1.7. Prove that for any sets A and B we have[3]

$$A \subset B, \quad \text{iff} \quad A \cap B = A, \quad \text{iff} \quad A \cup B = B.$$

[2]To make formulas clearer, sometimes we slightly abuse the notation and instead of, say, $A \cup \{x\}$, where x is an element outside A, we write just $A \cup x$. The same agreement holds true for other set-theoretic operations.

[3]Here, as usual, *iff* stands for "if and only if".

1.J Associativity of $\cap$ and $\cup$. For any sets A, B, and C, we have

$$(A \cap B) \cap C = A \cap (B \cap C) \quad \text{and} \quad (A \cup B) \cup C = A \cup (B \cup C).$$

Associativity allows us to not care about brackets and sometimes even to omit them. We define $A \cap B \cap C = (A \cap B) \cap C = A \cap (B \cap C)$ and $A \cup B \cup C = (A \cup B) \cup C = A \cup (B \cup C)$. However, the intersection and union of an arbitrarily large (in particular, infinite) collection of sets can be defined directly, without reference to the intersection or union of two sets. Indeed, let Γ be a collection of sets. The *intersection* of the sets in Γ is the set formed by the elements that belong to *every* set in Γ. This set is denoted by $\bigcap_{A \in \Gamma} A$. Similarly, the *union* of the sets in Γ is the set formed by elements that belong to *at least one* of the sets in Γ. This set is denoted by $\bigcup_{A \in \Gamma} A$.

1.K. The notions of intersection and union of an arbitrary collection of sets generalize the notions of intersection and union of two sets: for $\Gamma = \{A, B\}$, we have

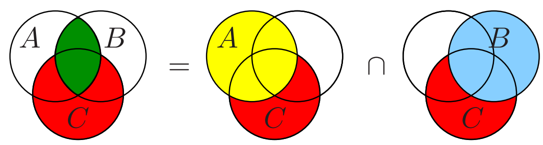
$$\bigcap_{C \in \Gamma} C = A \cap B \quad \text{and} \quad \bigcup_{C \in \Gamma} C = A \cup B.$$

1.8. Riddle. How are the notions of system of equations and intersection of sets related to each other?

1.L Two Distributivities. *For any sets A, B, and C, we have*

$$(A \cap B) \cup C = (A \cup C) \cap (B \cup C), \tag{1}$$

$$(A \cup B) \cap C = (A \cap C) \cup (B \cap C). \tag{2}$$



$(A \cap B) \cup C \quad = \quad (A \cup C) \quad \cap \quad (B \cup C)$

Figure 2. The left-hand side $(A \cap B) \cup C$ of equality (1) and the sets $A \cup C$ and $B \cup C$, whose intersection is the right-hand side of the equality (1).

1.M. Draw a Venn diagram illustrating (2). Prove (1) and (2) by tracing all details of the proofs in the Venn diagrams. Draw Venn diagrams illustrating all formulas below in this section.

1.9. Riddle. Generalize Theorem *1.L* to the case of arbitrary collections of sets.

1.N Yet Another Pair of Distributivities. *Let A be a set and let Γ be a set consisting of sets. Then we have*

$$A \cap \bigcup_{B\in\Gamma} B = \bigcup_{B\in\Gamma} (A \cap B) \quad \text{and} \quad A \cup \bigcap_{B\in\Gamma} B = \bigcap_{B\in\Gamma} (A \cup B).$$

⌈1′12⌋ Different Differences

The *difference* $A \smallsetminus B$ of two sets A and B is the set of those elements of A which do not belong to B. Here we do not assume that $A \supset B$.

If $A \supset B$, then the set $A \smallsetminus B$ is also called the *complement* of B in A.

1.10. Prove that for any sets A and B their union $A \cup B$ is the union of the following three sets: $A \smallsetminus B$, $B \smallsetminus A$, and $A \cap B$, which are pairwise disjoint.

1.11. Prove that $A \smallsetminus (A \smallsetminus B) = A \cap B$ for any sets A and B.

1.12. Prove that $A \subset B$ if and only if $A \smallsetminus B = \varnothing$.

1.13. Prove that $A \cap (B \smallsetminus C) = (A \cap B) \smallsetminus (A \cap C)$ for any sets A, B, and C.

The set $(A \smallsetminus B) \cup (B \smallsetminus A)$ is the *symmetric difference* of the sets A and B. It is denoted by $A \bigtriangleup B$.

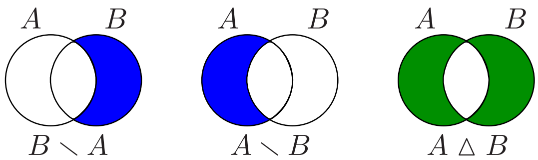


Figure 3. Differences of the sets A and B.

1.14. Prove that for any sets A and B we have

$$A \bigtriangleup B = (A \cup B) \smallsetminus (A \cap B).$$

1.15 Associativity of Symmetric Difference. Prove that for any sets A, B, and C we have

$$(A \bigtriangleup B) \bigtriangleup C = A \bigtriangleup (B \bigtriangleup C).$$

1.16. Riddle. Find a symmetric definition of the symmetric difference $(A \bigtriangleup B) \bigtriangleup C$ of three sets and generalize it to arbitrary finite collections of sets.

1.17 Distributivity. Prove that $(A \bigtriangleup B) \cap C = (A \cap C) \bigtriangleup (B \cap C)$ for any sets A, B, and C.

1.18. Does the following equality hold true for any sets A, B, and C:

$$(A \bigtriangleup B) \cup C = (A \cup C) \bigtriangleup (B \cup C)?$$

2. Topology on a Set

⌈2′1⌋ Definition of Topological Space

Let X be a set. Let Ω be a collection of its subsets such that:

(1) the union of any collection of sets that are elements of Ω belongs to Ω;

(2) the intersection of any finite collection of sets that are elements of Ω belongs to Ω;

(3) the empty set $\varnothing$ and the whole X belong to Ω.

Then

- Ω is a *topological structure* or just a *topology*[4] on X;
- the pair (X, Ω) is a *topological space*;
- elements of X are *points* of this topological space;
- elements of Ω are *open sets* of the topological space (X, Ω).

The conditions in the definition above are the *axioms of topological structure*.

⌈2′2⌋ Simplest Examples

A *discrete topological space* is a set with the topological structure consisting of all subsets.

2.A. Check that this is a topological space, i.e., all axioms of topological structure hold true.

An *indiscrete topological space* is the opposite example, in which the topological structure is the most meager. (It is also called *trivial topology*.) It consists only of X and $\varnothing$.

2.B. This is a topological structure, is it not?

Here are slightly less trivial examples.

2.1. Let X be the ray $[0, +\infty)$, and let Ω consist of $\varnothing$, X, and all rays $(a, +\infty)$ with $a \geq 0$. Prove that Ω is a topological structure.

2.2. Let X be a plane. Let Σ consist of $\varnothing$, X, and all open disks centered at the origin. Is Σ a topological structure?

2.3. Let X consist of four elements: $X = \{a, b, c, d\}$. Which of the following collections of its subsets are topological structures in X, i.e., satisfy the axioms of topological structure:

[4] Thus, Ω is important: it is called by the same word as the whole branch of mathematics. Certainly, this does not mean that Ω coincides with the subject of topology, but nearly everything in this subject is related to Ω.

(1) $\varnothing$, X, $\{a\}$, $\{b\}$, $\{a,c\}$, $\{a,b,c\}$, $\{a,b\}$;
(2) $\varnothing$, X, $\{a\}$, $\{b\}$, $\{a,b\}$, $\{b,d\}$;
(3) $\varnothing$, X, $\{a,c,d\}$, $\{b,c,d\}$?

The space of Problem *2.1* is the ***arrow***. We denote the space of Problem *2.3* (1) by . It is a sort of toy space made of 4 points. (The meaning of the pictogram is explained below in Section 7′9.) Both spaces, as well as the space of Problem *2.2*, are not very important, but they provide nice simple examples.

⌈2′3⌋ The Most Important Example: Real Line

Let X be the set $\mathbb{R}$ of all real numbers, Ω the set of arbitrary unions of open intervals (a, b) with $a, b \in \mathbb{R}$.

2.C. Check whether Ω satisfies the axioms of topological structure.

This is the topological structure which is always meant when $\mathbb{R}$ is considered as a topological space (unless another topological structure is explicitly specified). This space is usually called the ***real line***, and the structure is referred to as the ***canonical*** or ***standard*** topology on $\mathbb{R}$.

⌈2′4⌋ Additional Examples

2.4. Let X be $\mathbb{R}$, and let Ω consist of the empty set and all infinite subsets of $\mathbb{R}$. Is Ω a topological structure?

2.5. Let X be $\mathbb{R}$, and let Ω consists of the empty set and complements of all finite subsets of $\mathbb{R}$. Is Ω a topological structure?

The space of Problem *2.5* is denoted by $\mathbb{R}_{T_1}$ and called the ***line with*** T_1***-topology***.

2.6. Let (X, Ω) be a topological space, Y the set obtained from X by adding a single element a. Is

$$\{\{a\} \cup U \mid U \in \Omega\} \cup \{\varnothing\}$$

a topological structure in Y?

2.7. Is the set $\{\varnothing, \{0\}, \{0,1\}\}$ a topological structure in $\{0,1\}$?

If the topology Ω in Problem *2.6* is discrete, then the topology on Y is called a ***particular point topology*** or ***topology of everywhere dense point***. The topology in Problem *2.7* is a particular point topology; it is also called the ***topology of a connected pair of points*** or ***Sierpiński topology***.

2.8. List all topological structures in a two-element set, say, in $\{0,1\}$.

⌈2′5⌋ Using New Words: Points, Open Sets, Closed Sets

We recall that, for a topological space (X, Ω), elements of X are *points*, and elements of Ω are *open sets*.[5]

2.D. Reformulate the axioms of topological structure using the words *open set* wherever possible.

A set $F \subset X$ is *closed* in the space (X, Ω) if its complement $X \smallsetminus F$ is open (i.e., $X \smallsetminus F \in \Omega$).

⌈2′6⌋ Set-Theoretic Digression: De Morgan Formulas

2.E. Let Γ be an arbitrary collection of subsets of a set X. Then

$$X \smallsetminus \bigcup_{A \in \Gamma} A = \bigcap_{A \in \Gamma} (X \smallsetminus A), \tag{3}$$

$$X \smallsetminus \bigcap_{A \in \Gamma} A = \bigcup_{A \in \Gamma} (X \smallsetminus A). \tag{4}$$

Formula (4) is deduced from (3) in one step, is it not? These formulas are nonsymmetric cases of a single formulation, which contains, in a symmetric way, sets and their complements, unions, and intersections.

2.9. Riddle. Find such a formulation.

⌈2′7⌋ Properties of Closed Sets

2.F. Prove that:

(1) the intersection of any collection of closed sets is closed;
(2) the union of any finite number of closed sets is closed;
(3) the empty set and the whole space (i.e., the underlying set of the topological structure) are closed.

⌈2′8⌋ Being Open or Closed

Notice that the property of being closed is not the negation of the property of being open. (They are not exact antonyms in everyday usage, too.)

2.G. Find examples of sets that are

(1) both open and closed simultaneously (open-closed);
(2) neither open, nor closed.

[5]The letter Ω stands for the letter O which is the initial of the words with the same meaning: *Open* in English, *Otkrytyj* in Russian, *Offen* in German, *Ouvert* in French.

2.10. Give an explicit description of closed sets in
(1) a discrete space; (2) an indiscrete space;
(3) the arrow; (4) $\vee$;
(5) $\mathbb{R}_{T_1}$.

2.H. Is a closed segment $[a, b]$ closed in $\mathbb{R}$?

The concepts of closed and open sets are similar in a number of ways. The main difference is that the intersection of an infinite collection of open sets is not necessarily open, while the intersection of any collection of closed sets is closed. Along the same lines, the union of an infinite collection of closed sets is not necessarily closed, while the union of any collection of open sets is open.

2.11. Prove that the half-open interval $[0, 1)$ is neither open nor closed in $\mathbb{R}$, but is both a union of closed sets and an intersection of open sets.

2.12. Prove that the set $A = \{0\} \cup \{1/n \mid n \in \mathbb{N}\}$ is closed in $\mathbb{R}$.

⌈2′9⌋ Characterization of Topology in Terms of Closed Sets

2.13. Suppose a collection $\mathcal{F}$ of subsets of X satisfies the following conditions:

(1) the intersection of any family of sets from $\mathcal{F}$ belongs to $\mathcal{F}$;
(2) the union of any finite number sets from $\mathcal{F}$ belongs to $\mathcal{F}$;
(3) $\varnothing$ and X belong to $\mathcal{F}$.

Prove that then $\mathcal{F}$ is the set of all closed sets of a topological structure (which one?).

2.14. List all collections of subsets of a three-element set such that there are topologies where these collections are complete sets of closed sets.

⌈2′10⌋ Neighborhoods

A *neighborhood* of a point in a topological space is any open set containing this point. Analysts and French mathematicians (following N. Bourbaki) prefer a wider notion of neighborhood: they use this word for any set containing a neighborhood in the above sense.

2.15. Give an explicit description of all neighborhoods of a point in
(1) a discrete space; (2) an indiscrete space;
(3) the arrow; (4) $\vee$;
(5) a connected pair of points; (6) particular point topology.

⌈2′11x⌋ Open Sets on Line

2.Ix. *Prove that every open subset of the real line is a union of disjoint open intervals.*

At first glance, Theorem *2.Ix* suggests that open sets on the line are simple. However, an open set may lie on the line in a quite complicated manner. Its complement may happen to be not that simple. The complement of an open set is a closed set. One can naively expect that a closed set on $\mathbb{R}$ is a union of closed intervals. The next important example shows that this is very far from being true.

$\lceil 2'12\mathrm{x} \rfloor$ Cantor Set

Let K be the set of real numbers that are sums of series of the form $\sum_{k=1}^{\infty} a_k/3^k$ with $a_k \in \{0, 2\}$.

In other words, K consists of the real numbers that have the form $0.a_1a_2 \dots a_k \dots$ without the digit 1 in the number system with base 3.

***2.J*x.** Find a geometric description of K.

2.J*x*.1. Prove that

(1) K is contained in $[0, 1]$,
(2) K does not meet $(1/3, 2/3)$,
(3) K does not meet $\left(\frac{3s+1}{3^k}, \frac{3s+2}{3^k}\right)$ for any integers k and s.

2.J*x*.2. Present K as $[0, 1]$ with an infinite family of open intervals removed.

2.J*x*.3. Try to sketch K.

The set K is the *Cantor set*. It has a lot of remarkable properties and is involved in numerous problems below.

***2.K*x.** Prove that K is a closed set in the real line.

$\lceil 2'13\mathrm{x} \rfloor$ Topology and Arithmetic Progressions

***2.L*x*.** Consider the following property of a subset F of the set $\mathbb{N}$ of positive integers: there is $n \in \mathbb{N}$ such that F contains no arithmetic progressions of length n. Prove that subsets with this property together with the whole $\mathbb{N}$ form a collection of closed subsets in some topology on $\mathbb{N}$.

When solving this problem, you probably will need the following combinatorial theorem.

2.M*x *Van der Waerden's Theorem*. For every $n \in \mathbb{N}$, there is $N \in \mathbb{N}$ such that for any subset $A \subset \{1, 2, \dots, N\}$, either A or $\{1, 2, \dots, N\} \smallsetminus A$ contains an arithmetic progression of length n.

See [**3**].

3. Bases

⌈3′1⌋ Definition of Base

The topological structure is usually presented by describing its part, which is sufficient to recover the whole structure. A collection Σ of open sets is a *base* for a topology if each nonempty open set is a union of sets in Σ. For instance, all intervals form a base for the real line.

3.1. Can two distinct topological structures have the same base?

3.2. Find some bases for the topology of
(1) a discrete space; (2) ;
(3) an indiscrete space; (4) the arrow.
Try to choose the smallest possible bases.

3.3. Prove that any base of the canonical topology on $\mathbb{R}$ can be decreased.

3.4. Riddle. What topological structures have exactly one base?

⌈3′2⌋ When a Collection of Sets is a Base

3.A. *A collection Σ of open sets is a base for the topology iff for every open set U and every point $x \in U$ there is a set $V \in \Sigma$ such that $x \in V \subset U$.*

3.B. *A collection Σ of subsets of a set X is a base for a certain topology on X iff X is the union of all sets in Σ and the intersection of any two sets in Σ is the union of some sets in Σ.*

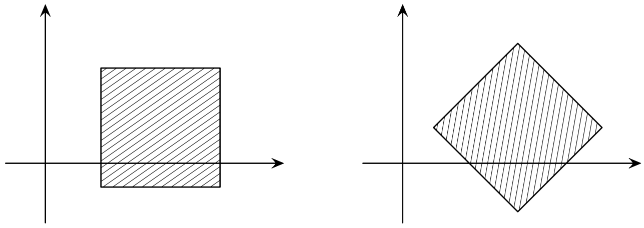
3.C. Show that the second condition in Theorem *3.B* (on the intersection) is equivalent to the following one: the intersection of any two sets in Σ contains, together with any of its points, a certain set in Σ containing this point (cf. Theorem *3.A*).

⌈3′3⌋ Bases for Plane

Consider the following three collections of subsets of $\mathbb{R}^2$:

- Σ^2, which consists of all possible open disks (i.e., disks without their boundary circles);
- Σ^∞, which consists of all possible open squares (i.e., squares without their sides and vertices) with sides parallel to the coordinate axes;
- Σ^1, which consists of all possible open squares with sides parallel to the bisectors of the coordinate angles.

(The squares in Σ^∞ and Σ^1 are determined by the inequalities $\max\{|x-a|, |y-b|\} < \rho$ and $|x-a|+|y-b| < \rho$, respectively.)



3.5. Prove that every element of Σ^2 is a union of elements of Σ^∞.

3.6. Prove that the intersection of any two elements of Σ^1 is a union of elements of Σ^1.

3.7. Prove that each of the collections Σ^2, Σ^∞, and Σ^1 is a base for some topological structure in $\mathbb{R}^2$, and that the structures determined by these collections coincide.

⌈3′4⌋ Subbases

Let (X, Ω) be a topological space. A collection Δ of its open subsets is a *subbase* for Ω provided that the collection

$$\Sigma = \{V \mid V = \bigcap_{i=1}^{k} W_i,\ k \in \mathbb{N},\ W_i \in \Delta\}$$

of all finite intersections of sets in Δ is a base for Ω.

3.8. Let X be a set, Δ a collection of subsets of X. Prove that Δ is a subbase for a topology on X iff $X = \bigcup_{W \in \Delta} W$.

⌈3′5⌋ Infiniteness of the Set of Prime Numbers

3.9. Prove that all (infinite) arithmetic progressions consisting of positive integers form a base for some topology on $\mathbb{N}$.

3.10. Using this topology, prove that the set of all prime numbers is infinite.

⌈3′6⌋ Hierarchy of Topologies

If Ω_1 and Ω_2 are topological structures in a set X such that $\Omega_1 \subset \Omega_2$, then Ω_2 is *finer* than Ω_1, and Ω_1 is *coarser* than Ω_2. For instance, the indiscrete topology is the coarsest topology among all topological structures in the same set, while the discrete topology is the finest one, is it not?

3.11. Show that the T_1-topology on the real line (see 2′4) is coarser than the canonical topology.

Two bases determining the same topological structure are *equivalent*.

3.D. Riddle. Formulate a necessary and sufficient condition for two bases to be equivalent without explicitly mentioning the topological structures determined by the bases. (Cf. *3.7*: the bases Σ^2, Σ^∞, and Σ^1 must satisfy the condition you are looking for.)

4. Metric Spaces

⌈4′1⌋ Definition and First Examples

A function[6] $\rho : X \times X \to \mathbb{R}_+ = \{ x \in \mathbb{R} \mid x \geq 0 \}$ is a *metric* (or *distance function*) on X if

(1) $\rho(x, y) = 0$ iff $x = y$;

(2) $\rho(x, y) = \rho(y, x)$ for any $x, y \in X$;

(3) $\rho(x, y) \leq \rho(x, z) + \rho(z, y)$ for any $x, y, z \in X$.

The pair (X, ρ), where ρ is a metric on X, is a *metric space*. Condition (3) is the *triangle inequality*.

4.A. Prove that the function

$$\rho : X \times X \to \mathbb{R}_+ : \ (x, y) \mapsto \begin{cases} 0 & \text{if } x = y, \\ 1 & \text{if } x \neq y \end{cases}$$

is a metric for any set X.

4.B. Prove that $\mathbb{R} \times \mathbb{R} \to \mathbb{R}_+ : (x, y) \mapsto |x - y|$ is a metric.

4.C. Prove that $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_+ : (x, y) \mapsto \sqrt{\sum_{i=1}^n (x_i - y_i)^2}$ is a metric.

The metrics of Problems *4.B* and *4.C* are always meant when $\mathbb{R}$ and $\mathbb{R}^n$ are considered as metric spaces, unless another metric is specified explicitly. The metric of Problem *4.B* is a special case of the metric of Problem *4.C*. All these metrics are called *Euclidean*.

⌈4′2⌋ Further Examples

4.1. Prove that $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_+ : (x, y) \mapsto \max_{i=1,\dots,n} |x_i - y_i|$ is a metric.

4.2. Prove that $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_+ : (x, y) \mapsto \sum_{i=1}^n |x_i - y_i|$ is a metric.

The metrics in $\mathbb{R}^n$ introduced in Problems *4.C*, *4.1*, *4.2* are members of an infinite sequence of metrics:

$$\rho^{(p)} : \ (x, y) \mapsto \left(\sum_{i=1}^n |x_i - y_i|^p \right)^{1/p}, \quad p \geq 1.$$

4.3. Prove that $\rho^{(p)}$ is a metric for any $p \geq 1$.

[6]The notions of function (mapping) and Cartesian square, as well as the corresponding notation, are discussed in detail below, in Sections 9 and 20. Nevertheless, we hope that the reader is acquainted with them, so we use them in this section without special explanations.

4.3.1 Hölder Inequality. Let $x_1, \dots, x_n, y_1, \dots, y_n \geq 0$, let $p, q > 0$, and let $1/p + 1/q = 1$. Prove that

$$\sum_{i=1}^{n} x_i y_i \leq \left(\sum_{i=1}^{n} x_i^p \right)^{1/p} \left(\sum_{i=1}^{n} y_i^q \right)^{1/q}.$$

The metric of *4.C* is $\rho^{(2)}$, that of *4.2* is $\rho^{(1)}$, and that of *4.1* can be denoted by $\rho^{(\infty)}$ and appended to the series since

$$\lim_{p \to +\infty} \left(\sum_{i=1}^{n} a_i^p \right)^{1/p} = \max a_i$$

for any positive $a_1, a_2, \dots, a_n$.

4.4. Riddle. How is this related to Σ^2, Σ^∞, and Σ^1 from Section 3?

For a real $p \geq 1$, denote by $l^{(p)}$ the set of sequences $x = \{x_i\}_{i=1,2,\dots}$ such that the series $\sum_{i=1}^{\infty} |x|^p$ converges.

4.5. Let $p \geq 1$. Prove that for any two sequences $x, y \in l^{(p)}$ the series $\sum_{i=1}^{\infty} |x_i - y_i|^p$ converges and that

$$(x, y) \mapsto \left(\sum_{i=1}^{\infty} |x_i - y_i|^p \right)^{1/p}$$

is a metric on $l^{(p)}$.

⌈4′3⌋ Balls and Spheres

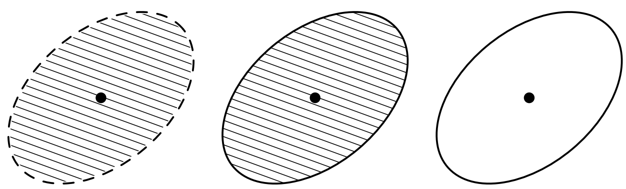
Let (X, ρ) be a metric space, $a \in X$ a point, r a positive real number. Then the sets

$$B_r(a) = \{ x \in X \mid \rho(a, x) < r \}, \tag{5}$$

$$D_r(a) = \{ x \in X \mid \rho(a, x) \leq r \}, \tag{6}$$

$$S_r(a) = \{ x \in X \mid \rho(a, x) = r \} \tag{7}$$

are, respectively, the *open ball*, *closed ball* (or *disk*), and *sphere* of the space (X, ρ) with center a and radius r.



⌈4′4⌋ Subspaces of a Metric Space

If (X, ρ) is a metric space and $A \subset X$, then the restriction of the metric ρ to $A \times A$ is a metric on A, and so $(A, \rho|_{A \times A})$ is a metric space. It is called a *subspace* of (X, ρ).

The disk $D_1(0)$ and the sphere $S_1(0)$ in $\mathbb{R}^n$ (with Euclidean metric, see *4.C*) are denoted by D^n and S^{n-1} and called the (*unit*) n-*disk* and $(n-1)$-*sphere*. They are regarded as metric spaces (with the metric induced from $\mathbb{R}^n$).

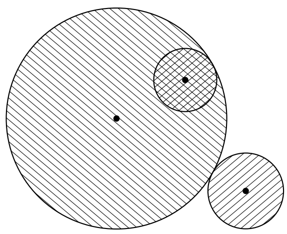
4.D. Check that D^1 is the segment $[-1, 1]$, D^2 is a plane disk, S^0 is the pair of points $\{-1, 1\}$, S^1 is a circle, S^2 is a sphere, and D^3 is a ball.

The last two assertions clarify the origin of the terms *sphere* and *ball* (in the context of metric spaces).

Some properties of balls and spheres in an arbitrary metric space resemble familiar properties of planar disks and circles and spatial balls and spheres.

4.E. Prove that for any points x and a of any metric space and any $r > \rho(a, x)$ we have

$$B_{r-\rho(a,x)}(x) \subset B_r(a) \text{ and } D_{r-\rho(a,x)}(x) \subset D_r(a).$$



4.6. Riddle. What if $r < \rho(x, a)$? What is an analog for the statement of Problem *4.E* in this case?

⌈4′5⌋ Surprising Balls

However, balls and spheres in other metric spaces may have rather surprising properties.

4.7. What are balls and spheres in $\mathbb{R}^2$ equipped with the metrics of *4.1* and *4.2*? (Cf. *4.4*.)

4.8. Find $D_1(a)$, $D_{1/2}(a)$, and $S_{1/2}(a)$ in the space of *4.A*.

4.9. Find a metric space and two balls in it such that the ball with the smaller radius contains the ball with the bigger one and does not coincide with it.

4.10. What is the minimal number of points in the space which is required to be constructed in *4.9*?

4.11. Prove that the largest radius in *4.9* is at most twice the smaller radius.

⌈4′6⌋ Segments (What Is Between)

4.12. Prove that the segment with endpoints $a, b \in \mathbb{R}^n$ can be described as

$$\{\, x \in \mathbb{R}^n \mid \rho(a,x) + \rho(x,b) = \rho(a,b) \,\},$$

where ρ is the Euclidean metric.

4.13. How does the set defined as in Problem *4.12* look if ρ is the metric defined in Problems *4.1* or *4.2*? (Consider the case where $n = 2$ if it seems to be easier.)

⌈4′7⌋ Bounded Sets and Balls

A subset A of a metric space (X, ρ) is *bounded* if there is a number $d > 0$ such that $\rho(x,y) < d$ for any $x, y \in A$. The greatest lower bound for such d is the *diameter* of A. It is denoted by $\operatorname{diam}(A)$.

4.F. Prove that a set A is bounded iff A is contained in a ball.

4.14. What is the relation between the minimal radius of such a ball and $\operatorname{diam}(A)$?

⌈4′8⌋ Norms and Normed Spaces

Let X be a vector space (over $\mathbb{R}$). A function $X \to \mathbb{R}_+ : x \mapsto \|x\|$ is a *norm* if

(1) $\|x\| = 0$ iff $x = 0$;
(2) $\|\lambda x\| = |\lambda| \|x\|$ for any $\lambda \in \mathbb{R}$ and $x \in X$;
(3) $\|x + y\| \le \|x\| + \|y\|$ for any $x, y \in X$.

4.15. Prove that if $x \mapsto \|x\|$ is a norm, then

$$\rho : X \times X \to \mathbb{R}_+ : (x,y) \mapsto \|x - y\|$$

is a metric.

A vector space equipped with a norm is a *normed space*. The metric determined by the norm as in *4.15* transforms the normed space into a metric space in a canonical way.

4.16. Look through the problems of this section and figure out which of the metric spaces involved are, in fact, normed vector spaces.

4.17. Prove that every ball in a normed space is a convex[7] set symmetric with respect to the center of the ball.

4.18*. Prove that every convex closed bounded set in $\mathbb{R}^n$ that has a center of symmetry and is not contained in any affine space except $\mathbb{R}^n$ itself is a unit ball with respect to a certain norm, which is uniquely determined by this ball.

---

[7]Recall that a set A is *convex* if for any $x, y \in A$ the segment connecting x and y is contained in A. Certainly, this definition involves the notion of *segment*, so it makes sense only for subsets of those spaces where the notion of segment connecting two points makes sense. This is the case in vector and affine spaces over $\mathbb{R}$.

⌈4′9⌋ Metric Topology

4.G. *The collection of all open balls in the metric space is a base for a certain topology.*

This topology is the *metric topology*. We also say that it is *generated* by the metric. This topological structure is always meant whenever the metric space is regarded as a topological space (for instance, when we speak about open and closed sets, neighborhoods, etc. in this space).

4.H. Prove that the standard topological structure in $\mathbb{R}$ introduced in Section 2 is generated by the metric $(x, y) \mapsto |x - y|$.

4.19. What topological structure is generated by the metric of *4.A*?

4.I. *A set U is open in a metric space iff, together with each of its points, the set U contains a ball centered at this point.*

⌈4′10⌋ Openness and Closedness of Balls and Spheres

4.20. Prove that a closed ball is closed (here and below, we mean the metric topology).

4.21. Find a closed ball that is open.

4.22. Find an open ball that is closed.

4.23. Prove that a sphere is closed.

4.24. Find a sphere that is open.

⌈4′11⌋ Metrizable Topological Spaces

A topological space is *metrizable* if its topological structure is generated by a certain metric.

4.J. An indiscrete space is not metrizable if it is not a singleton (otherwise, it has too few open sets).

4.K. A finite space X is metrizable iff it is discrete.

4.25. Which of the topological spaces described in Section 2 are metrizable?

⌈4′12⌋ Equivalent Metrics

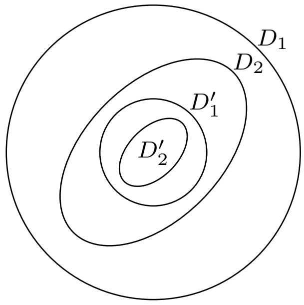
Two metrics in the same set are *equivalent* if they generate the same topology.

4.26. Are the metrics of *4.C*, *4.1*, and *4.2* equivalent?

4.27. Prove that two metrics ρ_1 and ρ_2 in X are equivalent if there are numbers $c, C > 0$ such that

$$c\rho_1(x, y) \leq \rho_2(x, y) \leq C\rho_1(x, y)$$

for any $x, y \in X$.



4.28. Generally speaking, the converse is not true.

4.29. Riddle. Hence, the condition of equivalence of metrics formulated in Problem *4.27* can be weakened. How?

4.30. The metrics $\rho^{(p)}$ in $\mathbb{R}^n$ defined right before Problem *4.3* are equivalent.

4.31*. Prove that the following two metrics ρ_1 and ρ_C in the set of all continuous functions $[0,1] \to \mathbb{R}$ are not equivalent:

$$\rho_1(f,g) = \int_0^1 \big|f(x) - g(x)\big| dx, \qquad \rho_C(f,g) = \max_{x\in[0,1]} \big|f(x) - g(x)\big|.$$

Is it true that one of the topological structures generated by them is finer than the other one?

⌈4′13⌋ Operations with Metrics

4.32. 1) Prove that if ρ_1 and ρ_2 are two metrics in X, then $\rho_1+\rho_2$ and $\max\{\rho_1,\rho_2\}$ also are metrics. 2) Are the functions $\min\{\rho_1,\rho_2\}$, $\rho_1\rho_2$, and ρ_1/ρ_2 metrics? (By definition, for $\rho = \rho_1/\rho_2$ we put $\rho(x,x) = 0$.)

4.33. Prove that if $\rho : X \times X \to \mathbb{R}_+$ is a metric, then

(1) the function $(x,y) \mapsto \dfrac{\rho(x,y)}{1+\rho(x,y)}$ is a metric;
(2) the function $(x,y) \mapsto \min\{\rho(x,y), 1\}$ is a metric;
(3) the function $(x,y) \mapsto f\big(\rho(x,y)\big)$ is a metric if f satisfies the following conditions:
  (a) $f(0) = 0$,
  (b) f is a monotone increasing function, and
  (c) $f(x+y) \le f(x) + f(y)$ for any $x, y \in \mathbb{R}$.

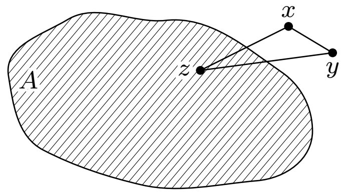
4.34. Prove that the metrics ρ and $\dfrac{\rho}{1+\rho}$ are equivalent.

⌈4′14⌋ Distances between Points and Sets

Let (X,ρ) be a metric space, $A \subset X$, and $b \in X$. The number $\rho(b,A) = \inf\{\, \rho(b,a) \mid a \in A \,\}$ is the *distance from the point b to the set A*.

4.L. Let A be a closed set. Prove that $\rho(b,A) = 0$ iff $b \in A$.

4.35. Prove that $|\rho(x,A) - \rho(y,A)| \le \rho(x,y)$ for any set A and any points x and y in a metric space.



$\rho(x, A) \leqslant \rho(x, z) \leqslant \rho(x, y)+\rho(y, z)$

⌈4′15x⌋ Distance between Sets

Let A and B be two bounded subsets in a metric space (X, ρ). We define

$$d_\rho(A, B) = \max\Big\{\sup_{a\in A} \rho(a, B),\ \sup_{b\in B} \rho(b, A)\Big\}.$$

This number is the *Hausdorff distance* between A and B.

4.Mx. Prove that the Hausdorff distance between bounded subsets of a metric space satisfies conditions (2) and (3) in the definition of a metric.

4.Nx. Prove that for every metric space the Hausdorff distance is a metric on the set of its closed bounded subsets.

Let A and B be two bounded polygons in the plane.[8] We define

$$d_\triangle(A, B) = S(A) + S(B) - 2S(A \cap B),$$

where $S(C)$ is the area of a polygon C.

4.Ox. Prove that $d_\triangle$ is a metric on the set of all bounded plane polygons.

We call d_Δ the *area metric*.

4.Px. Prove that the area metric is *not* equivalent to the Hausdorff metric on the set of all bounded plane polygons.

4.Qx. Prove that the area metric *is* equivalent to the Hausdorff metric on the set of *convex* bounded plane polygons.

⌈4′16x⌋ Ultrametrics and p-Adic Numbers

A metric ρ is an *ultrametric* if it satisfies the *ultrametric triangle inequality*:

$$\rho(x, y) \le \max\{\rho(x, z),\ \rho(z, y)\}$$

for any x, y, and z.

A metric space (X, ρ), where ρ is an ultrametric, is an *ultrametric space*.

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[8] Although we assume that the notion of a bounded polygon is well known from elementary geometry, nevertheless, we recall the definition. A *bounded plane polygon* is the set of the points of a simple closed polygonal line γ and the points surrounded by γ. A *simple closed polygonal line* (or *polyline*) is a cyclic sequence of segments each of which starts at the point where the previous one ends and these are the only pairwise intersections of the segments.

***4.R*x.** Check that only one metric in *4.A–4.2* is an ultrametric. Which one?

***4.S*x.** Prove that all triangles in an ultrametric space are isosceles (i.e., for any three points a, b, and c, at least two of the three distances $\rho(a,b)$, $\rho(b,c)$, and $\rho(a,c)$ are equal).

***4.T*x.** Prove that spheres in an ultrametric space are not only closed (see Problem *4.23*), but also open.

The most important example of an ultrametric is the *p-adic metric* in the set $\mathbb{Q}$ of rational numbers. Let p be a prime number. For $x, y \in \mathbb{Q}$, present the difference $x - y$ as $\frac{r}{s}p^{\alpha}$, where r, s, and α are integers, and r and s are co-prime with p. We define $\rho(x,y) = p^{-\alpha}$.

***4.U*x.** Prove that ρ is an ultrametric.

⌈4′17x⌋ Asymmetrics

A function $\rho : X \times X \to \mathbb{R}_+$ is an *asymmetric* on a set X if

(1) $\rho(x,y) = 0$ and $\rho(y,x) = 0$, iff $x = y$;

(2) $\rho(x,y) \le \rho(x,z) + \rho(z,y)$ for any $x, y, z \in X$.

Thus, an asymmetric satisfies conditions 1 and 3 in the definition of a metric, but, maybe, does not satisfy condition 2.

Here is example of an asymmetric taken "from real life": the length of the shortest path from one place to another by car in a city having one-way streets.

***4.V*x.** Prove that if $\rho : X \times X \to \mathbb{R}_+$ is an asymmetric, then the function

$$(x,y) \mapsto \rho(x,y) + \rho(y,x)$$

is a metric on X.

Let A and B be two bounded subsets of a metric space (X,ρ). The number $a_\rho(A,B) = \sup_{b\in B} \rho(b,A)$ is the *asymmetric distance from A to B*.

***4.W*x.** The function a_ρ on the set of bounded subsets of a metric space satisfies the triangle inequality in the definition of an asymmetric.

***4.X*x.** Let (X,ρ) be a metric space. A set $B \subset X$ is contained in all closed sets containing $A \subset X$ iff $a_\rho(A,B) = 0$.

***4.Y*x.** Prove that a_ρ is an asymmetric on the set of all bounded closed subsets of a metric space (X,ρ).

Let A and B be two polygons on the plane. We define

$$a_\Delta(A,B) = S(B) - S(A\cap B) = S(B \smallsetminus A),$$

where $S(C)$ is the area of a polygon C.

4.36x. Prove that $a_\triangle$ is an asymmetric on the set of all planar polygons.

A pair (X, ρ), where ρ is an asymmetric on X, is an *asymmetric space*. Certainly, any metric space is an asymmetric space, too. Open and closed balls and spheres in an asymmetric space are defined as in a metric space, see Section 4′3.

4.Zx. *The set of all open balls of an asymmetric space is a base of a certain topology.*

We also say that this topology is *generated* by the asymmetric.

4.37x. Prove that the formula $a(x, y) = \max\{x - y, 0\}$ determines an asymmetric on $[0, \infty)$, and the topology generated by this asymmetric is the arrow topology, see Section 2′2.

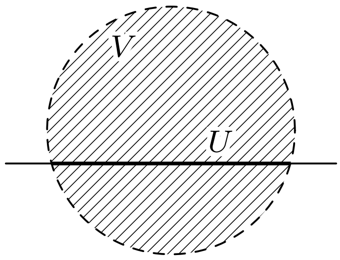
5. Subspaces

⌈5′1⌋ Topology for a Subset of a Space

Let (X, Ω) be a topological space, $A \subset X$. Denote by Ω_A the collection of sets $A \cap V$, where $V \in \Omega$: $\Omega_A = \{A \cap V \mid V \in \Omega\}$.

5.A. *The collection Ω_A is a topological structure in A.*

The pair (A, Ω_A) is a ***subspace*** of the space (X, Ω). The collection Ω_A is the ***subspace topology***, the ***relative topology***, or the topology ***induced*** on A by Ω, and its elements are said to be sets ***open*** in A.



5.B. The canonical topology on $\mathbb{R}^1$ coincides with the topology induced on $\mathbb{R}^1$ as on a subspace of $\mathbb{R}^2$.

5.1. Riddle. How to construct a base for the topology induced on A by using a base for the topology on X?

5.2. Describe the topological structures induced

(1) on the set $\mathbb{N}$ of positive integers by the topology of the real line;
(2) on $\mathbb{N}$ by the topology of the arrow;
(3) on the two-element set $\{1, 2\}$ by the topology of $\mathbb{R}_{T_1}$;
(4) on the same set by the topology of the arrow.

5.3. Is the half-open interval $[0, 1)$ open in the segment $[0, 2]$ regarded as a subspace of the real line?

5.C. *A set F is closed in a subspace $A \subset X$ iff F is the intersection of A and a closed subset of X.*

5.4. If a subset of a subspace is open (respectively, closed) in the ambient space, then it is also open (respectively, closed) in the subspace.

⌈5′2⌋ Relativity of Openness and Closedness

Sets that are open in a subspace are not necessarily open in the ambient space.

5.D. The unique open set in $\mathbb{R}^1$ which is also open in $\mathbb{R}^2$ is $\varnothing$.

However, the following is true.

5.E. An open set of an open subspace is open in the ambient space, i.e., if $A \in \Omega$, then $\Omega_A \subset \Omega$.

The same relation holds true for closed sets. Sets that are closed in the subspace are not necessarily closed in the ambient space. However, the following is true.

5.F. Closed sets of a closed subspace are closed in the ambient space.

5.5. Prove that a set U is open in X iff each point in U has a neighborhood V in X such that $U \cap V$ is open in V.

This allows us to say that the property of being open is *local*. Indeed, we can reformulate *5.5* as follows: a set is open iff it is open in a neighborhood of each of its points.

5.6. Show that the property of being closed is not local.

5.G Transitivity of Induced Topology. *Let (X, Ω) be a topological space, $X \supset A \supset B$. Then $(\Omega_A)_B = \Omega_B$, i.e., the topology induced on B by the relative topology of A coincides with the topology induced on B directly from X.*

5.7. Let (X, ρ) be a metric space, $A \subset X$. Then the topology on A generated by the induced metric $\rho|_{A \times A}$ coincides with the relative topology induced on A by the metric topology on X.

5.8. Riddle. The statement *5.7* is equivalent to a pair of inclusions. Which of them is less obvious?

⌈5′3⌋ Agreement on Notation for Topological Spaces

Different topological structures in the same set are considered simultaneously rather seldom. This is why a topological space is usually denoted by the same symbol as the set of its points, i.e., instead of (X, Ω) we write just X. The same applies to metric spaces: instead of (X, ρ) we write just X.

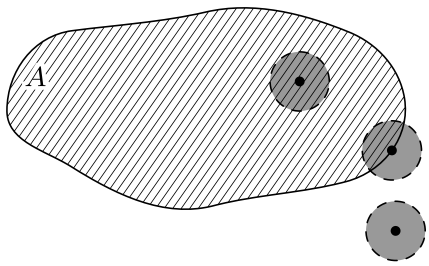
6. Position of a Point with Respect to a Set

This section is devoted to further expanding the vocabulary needed when we speak about phenomena in a topological space.

⌈6′1⌋ Interior, Exterior, and Boundary Points

Let X be a topological space, $A \subset X$ a subset, and $b \in X$ a point. The point b is

- an *interior* point of A if b has a neighborhood contained in A;
- an *exterior* point of A if b has a neighborhood disjoint with A;
- a *boundary* point of A if each neighborhood of b meets both A and the complement of A.



⌈6′2⌋ Interior and Exterior

The *interior* of a set A in a topological space X is the greatest (with respect to inclusion) open set in X contained in A, i.e., an open set that contains any other open subset of A. It is denoted by $\operatorname{Int} A$ or, in more detail, by $\operatorname{Int}_X A$.

6.A. *Every subset of a topological space has an interior. It is the union of all open sets contained in this set.*

6.B. *The interior of a set A is the set of interior points of A.*

6.C. *A set is open iff it coincides with its interior.*

6.D. Prove that in $\mathbb{R}$:

(1) $\operatorname{Int}[0,1) = (0,1)$,

(2) $\operatorname{Int}\mathbb{Q} = \varnothing$, and

(3) $\operatorname{Int}(\mathbb{R} \smallsetminus \mathbb{Q}) = \varnothing$.

6.1. Find the interior of $\{a, b, d\}$ in the space .

6.2. Find the interior of the interval $(0, 1)$ on the line with the Zariski topology.

The *exterior* of a set is the greatest open set disjoint with A. Obviously, the exterior of A is $\operatorname{Int}(X \smallsetminus A)$.

⌈6′3⌋ Closure

The *closure* of a set A is the smallest closed set containing A. It is denoted by $\operatorname{Cl} A$ or, more specifically, by $\operatorname{Cl}_X A$.

6.E. *Every subset of a topological space has a closure. It is the intersection of all closed sets containing this set.*

6.3. Prove that if A is a subspace of X and $B \subset A$, then $\operatorname{Cl}_A B = (\operatorname{Cl}_X B) \cap A$. Is it true that $\operatorname{Int}_A B = (\operatorname{Int}_X B) \cap A$?

A point b is an *adherent* point for a set A if all neighborhoods of b meet A.

6.F. *The closure of a set A is the set of the adherent points of A.*

6.G. *A set A is closed iff $A = \operatorname{Cl} A$.*

6.H. *The closure of a set A is the complement of the exterior of A. In formulas:* $\operatorname{Cl} A = X \smallsetminus \operatorname{Int}(X \smallsetminus A)$, *where X is the space and $A \subset X$.*

6.I. Prove that in $\mathbb{R}$ we have:

(1) $\operatorname{Cl}[0, 1) = [0, 1]$,
(2) $\operatorname{Cl} \mathbb{Q} = \mathbb{R}$, and
(3) $\operatorname{Cl}(\mathbb{R} \smallsetminus \mathbb{Q}) = \mathbb{R}$.

6.4. Find the closure of $\{a\}$ in .

⌈6′4⌋ Closure in Metric Space

Let A be a subset and b a point of a metric space (X, ρ). We recall that the distance $\rho(b, A)$ from b to A is $\inf\{\, \rho(b, a) \mid a \in A \,\}$ (see 4′14).

6.J. Prove that $b \in \operatorname{Cl} A$ iff $\rho(b, A) = 0$.

⌈6′5⌋ Boundary

The *boundary* of a set A is the set $\operatorname{Cl} A \smallsetminus \operatorname{Int} A$. It is denoted by $\operatorname{Fr} A$ or, in more detail, $\operatorname{Fr}_X A$.

6.5. Find the boundary of $\{a\}$ in .

6.K. The boundary of a set is the set of its boundary points.

6.L. Prove that a set A is closed iff $\operatorname{Fr} A \subset A$.

6.6. 1) Prove that $\operatorname{Fr} A = \operatorname{Fr}(X \smallsetminus A)$. 2) Find a formula for $\operatorname{Fr} A$ which is symmetric with respect to A and $X \smallsetminus A$.

6.7. The boundary of a set A equals the intersection of the closure of A and the closure of the complement of A: we have $\operatorname{Fr} A = \operatorname{Cl} A \cap \operatorname{Cl}(X \smallsetminus A)$.

⌈6′6⌋ Closure and Interior with Respect to a Finer Topology

6.8. Let Ω_1 and Ω_2 be two topological structures in X such that $\Omega_1 \subset \Omega_2$. Let Cl_i denote the closure with respect to Ω_i. Prove that $\operatorname{Cl}_1 A \supset \operatorname{Cl}_2 A$ for any $A \subset X$.

6.9. Formulate and prove a similar statement about the interior.

⌈6′7⌋ Properties of Interior and Closure

6.10. Prove that if $A \subset B$, then $\operatorname{Int} A \subset \operatorname{Int} B$.

6.11. Prove that $\operatorname{Int} \operatorname{Int} A = \operatorname{Int} A$.

6.12. Find out whether the following equalities hold true that for any sets A and B:

$$\operatorname{Int}(A \cap B) = \operatorname{Int} A \cap \operatorname{Int} B, \tag{8}$$

$$\operatorname{Int}(A \cup B) = \operatorname{Int} A \cup \operatorname{Int} B. \tag{9}$$

6.13. Give an example in which one of equalities (8) and (9) is wrong.

6.14. In the example that you found when solving Problem *6.12*, an inclusion of one side into another one holds true. Does this inclusion hold true for arbitrary A and B?

6.15. Study the operator Cl in a way suggested by the investigation of Int undertaken in *6.10–6.14*.

6.16. Find $\operatorname{Cl}\{1\}$, $\operatorname{Int}[0,1]$, and $\operatorname{Fr}(2,+\infty)$ in the arrow.

6.17. Find $\operatorname{Int}\big((0,1] \cup \{2\}\big)$, $\operatorname{Cl}\{\, 1/n \mid n \in \mathbb{N} \,\}$, and $\operatorname{Fr} \mathbb{Q}$ in $\mathbb{R}$.

6.18. Find $\operatorname{Cl} \mathbb{N}$, $\operatorname{Int}(0,1)$, and $\operatorname{Fr}[0,1]$ in $\mathbb{R}_{T_1}$. How do you find the closure and interior of a set in this space?

6.19. Does a sphere contain the boundary of the open ball with the same center and radius?

6.20. Does a sphere contain the boundary of the closed ball with the same center and radius?

6.21. Find an example in which a sphere is disjoint with the closure of the open ball with the same center and radius.

⌈6′8⌋ Compositions of Closure and Interior

6.22 Kuratowski's Problem. How many pairwise distinct sets can one obtain from of a single set by using the operators Cl and Int?

The following problems will help you to solve Problem *6.22*.

6.22.1. Find a set $A \subset \mathbb{R}$ such that the sets A, $\operatorname{Cl} A$, and $\operatorname{Int} A$ are pairwise distinct.

6.22.2. Is there a set $A \subset \mathbb{R}$ such that

(1) A, $\operatorname{Cl} A$, $\operatorname{Int} A$, and $\operatorname{Cl}\operatorname{Int} A$ are pairwise distinct;
(2) A, $\operatorname{Cl} A$, $\operatorname{Int} A$, and $\operatorname{Int}\operatorname{Cl} A$ are pairwise distinct;
(3) A, $\operatorname{Cl} A$, $\operatorname{Int} A$, $\operatorname{Cl}\operatorname{Int} A$, and $\operatorname{Int}\operatorname{Cl} A$ are pairwise distinct?

If you find such sets, keep on going in the same way, and when you fail to proceed, try to formulate a theorem explaining the failure.

6.22.3. Prove that $\operatorname{Cl}\operatorname{Int}\operatorname{Cl}\operatorname{Int} A = \operatorname{Cl}\operatorname{Int} A$.

⌈6′9⌋ Sets with Common Boundary

6.23*. Find three open sets in the real line that have the same boundary. Is it possible to increase the number of such sets?

⌈6′10⌋ Convexity and Int, Cl, and Fr

Recall that a set $A \subset \mathbb{R}^n$ is *convex* if together with any two points it contains the entire segment connecting them (i.e., for any $x, y \in A$, every point z of the segment $[x, y]$ belongs to A).

Let A be a convex set in $\mathbb{R}^n$.

6.24. Prove that $\operatorname{Cl} A$ and $\operatorname{Int} A$ are convex.

6.25. Prove that A contains a ball if A is not contained in an $(n-1)$-dimensional affine subspace of $\mathbb{R}^n$.

6.26. When is $\operatorname{Fr} A$ convex?

⌈6′11⌋ Characterization of Topology by Operations of Taking Closure and Interior

6.27*. Suppose that Cl_* is an operator on the set of all subsets of a set X, which has the following properties:

(1) $\mathrm{Cl}_* \varnothing = \varnothing$,
(2) $\mathrm{Cl}_* A \supset A$,
(3) $\mathrm{Cl}_*(A \cup B) = \mathrm{Cl}_* A \cup \mathrm{Cl}_* B$,
(4) $\mathrm{Cl}_* \mathrm{Cl}_* A = \mathrm{Cl}_* A$.

Prove that $\Omega = \{ U \subset X \mid \mathrm{Cl}_*(X \smallsetminus U) = X \smallsetminus U \}$ is a topological structure and $\mathrm{Cl}_* A$ is the closure of a set A in the space (X, Ω).

6.28. Present a similar system of axioms for Int.

⌈6′12⌋ Dense Sets

Let A and B be two sets in a topological space X. A is *dense in* B if $\operatorname{Cl} A \supset B$, and A is *everywhere dense* if $\operatorname{Cl} A = X$.

6.M. *A set is everywhere dense iff it meets any nonempty open set.*

6.N. The set $\mathbb{Q}$ is everywhere dense in $\mathbb{R}$.

6.29. Give an explicit characterization of everywhere dense sets 1) in an indiscrete space, 2) in the arrow, and 3) in $\mathbb{R}_{T_1}$.

6.30. Prove that a topological space is discrete iff it contains a unique everywhere dense set. (By the way, which one?)

6.31. Formulate a necessary and sufficient condition on the topology of a space which has an everywhere dense point. Find spaces in Section 2 that satisfy this condition.

6.32. 1) Is it true that the union of everywhere dense sets is everywhere dense? 2) Is it true that the intersection of two everywhere dense sets is everywhere dense?

6.33. Prove that any two open everywhere dense sets have everywhere dense intersection.

6.34. Which condition in Problem *6.33* is redundant?

6.35*. 1) Prove that a countable intersection of open everywhere dense sets in $\mathbb{R}$ is everywhere dense. 2) Is it possible to replace $\mathbb{R}$ here by an arbitrary topological space?

6.36*. Prove that $\mathbb{Q}$ is not the intersection of countably many open sets in $\mathbb{R}$.

⌈6′13⌋ Nowhere Dense Sets

A set is *nowhere dense* if its exterior is everywhere dense.

6.37. Can a set be everywhere dense and nowhere dense simultaneously?

6.O. A set A is nowhere dense in X iff each neighborhood of each point $x \in X$ contains a point y such that the complement of A contains y together with a neighborhood of y.

6.38. Riddle. What can you say about the interior of a nowhere dense set?

6.39. Is $\mathbb{R}$ nowhere dense in $\mathbb{R}^2$?

6.40. Prove that if A is nowhere dense, then $\operatorname{Int}\operatorname{Cl} A = \varnothing$.

6.41. 1) Prove that the boundary of a closed set is nowhere dense. 2) Is this true for the boundary of an open set? 3) Is this true for the boundary of an arbitrary set?

6.42. Prove that a finite union of nowhere dense sets is nowhere dense.

6.43. Prove that for every set A there exists a greatest open set B in which A is dense. The extreme cases $B = X$ and $B = \varnothing$ mean that A is either everywhere dense or nowhere dense, respectively.

6.44*. Prove that $\mathbb{R}$ is not the union of countably many nowhere-dense subsets.

⌈6′14⌋ Limit Points and Isolated Points

A point b is a *limit point* of a set A if each neighborhood of b meets $A \smallsetminus b$.

6.P. Every limit point of a set is its adherent point.

6.45. Present an example in which an adherent point is not a limit one.

A point b is an *isolated point* of a set A if $b \in A$ and b has a neighborhood disjoint with $A \smallsetminus b$.

6.Q. *A set A is closed iff A contains all of its limit points.*

6.46. Find limit and isolated points of the sets $(0, 1] \cup \{2\}$ and $\{\, 1/n \mid n \in \mathbb{N} \,\}$ in $\mathbb{Q}$ and in $\mathbb{R}$.

6.47. Find limit and isolated points of the set $\mathbb{N}$ in $\mathbb{R}_{T_1}$.

⌈6′15⌋ Locally Closed Sets

A subset A of a topological space X is *locally closed* if each point of A has a neighborhood U such that $A \cap U$ is closed in U (cf. Problems *5.5–5.6*).

6.48. Prove that the following conditions are equivalent:

(1) A is locally closed in X;
(2) A is an open subset of its closure $\operatorname{Cl} A$;
(3) A is the intersection of open and closed subsets of X.

7. Ordered Sets

This section is devoted to orders. They are structures on sets and occupy a position in Mathematics almost as profound as topological structures. After a short general introduction, we focus on relations between structures of these two types. Similarly to metric spaces, partially ordered sets possess natural topological structures. This is a source of interesting and important examples of topological spaces. As we will see later (in Section 21), practically all finite topological spaces appear in this way.

⌈7′1⌋ Strict Orders

A *binary relation* on a set X is a set of ordered pairs of elements of X, i.e., a subset $R \subset X \times X$. Many relations are denoted by special symbols, like $\prec$, $\vdash$, $\equiv$, or $\sim$. When such notation is used, there is a tradition to write xRy instead of writing $(x, y) \in R$. So, we write $x \vdash y$, or $x \sim y$, or $x \prec y$, etc. This generalizes the usual notation for the classical binary relations $=$, $<$, $>$, $\leq$, $\subset$, etc.

A binary relation $\prec$ on a set X is a *strict partial order*, or just a *strict order* if it satisfies the following two conditions:

- *Irreflexivity*: There is no $a \in X$ such that $a \prec a$.
- *Transitivity*: $a \prec b$ and $b \prec c$ imply $a \prec c$ for any $a, b, c \in X$.

7.A Antisymmetry. Let $\prec$ be a strict partial order on a set X. There are no $x, y \in X$ such that $x \prec y$ and $y \prec x$ simultaneously.

7.B. Relation $<$ in the set $\mathbb{R}$ of real numbers is a strict order.

The formula $a \prec b$ is sometimes read as "a is less than b" or "b is greater than a", but it is often read as "b follows a" or "a precedes b". The advantage of the latter two ways of reading is that then the relation $\prec$ is not associated too closely with the inequality between real numbers.

⌈7′2⌋ Nonstrict Orders

A binary relation $\preceq$ on a set X is a *nonstrict partial order*, or just a *nonstrict order*, if it satisfies the following three conditions:

- *Transitivity*: If $a \preceq b$ and $b \preceq c$, then $a \preceq c$ for any $a, b, c \in X$.
- *Antisymmetry*: If $a \preceq b$ and $b \preceq a$, then $a = b$ for any $a, b \in X$.
- *Reflexivity*: $a \preceq a$ for any $a \in X$.

7.C. The relation $\leq$ on $\mathbb{R}$ is a nonstrict order.

7.D. In the set $\mathbb{N}$ of positive integers, the relation $a \mid b$ (a ***divides*** b) is a nonstrict partial order.

> ***7.1.*** Is the relation $a \mid b$ a nonstrict partial order on the set $\mathbb{Z}$ of integers?

7.E. Inclusion determines a nonstrict partial order on the set of subsets of any set X.

⌈7′3⌋ Relation between Strict and Nonstrict Orders

7.F. For each strict order $\prec$, there is a relation $\preceq$ defined on the same set as follows: $a \preceq b$ if either $a \prec b$, or $a = b$. This relation is a nonstrict order.

The nonstrict order $\preceq$ of *7.F* is ***associated*** with the original strict order $\prec$.

7.G. For each nonstrict order $\preceq$, there is a relation $\prec$ defined on the same set as follows: $a \prec b$ if $a \preceq b$ and $a \neq b$. This relation is a strict order.

The strict order $\prec$ of *7.G* is ***associated*** with the original nonstrict order $\preceq$.

7.H. The constructions of Problems *7.F* and *7.G* are mutually inverse: applied one after another in any order, they give the initial relation.

Thus, strict and nonstrict orders determine each other. They are just different incarnations of the same structure of order. We have already met a similar phenomenon in topology: open and closed sets in a topological space determine each other and provide different ways for describing a topological structure.

A set equipped with a partial order (either strict or nonstrict) is a ***partially ordered set*** or, briefly, a ***poset***. More formally speaking, a partially ordered set is a pair $(X, \prec)$ formed by a set X and a strict partial order $\prec$ on X. Certainly, instead of a strict partial order $\prec$ we can use the corresponding nonstrict order $\preceq$.

Which of the orders, strict or nonstrict, prevails in each specific case is a matter of convenience, taste, and tradition. Although it would be handy to keep both of them available, nonstrict orders conquer situation by situation. For instance, nobody introduces special notation for strict divisibility. Another example: the symbol $\subseteq$, which is used to denote nonstrict inclusion, is replaced by the symbol $\subset$, which is almost never understood as a designation solely for strict inclusion.

In abstract considerations, we use both kinds of orders: strict partial orders are denoted by the symbol $\prec$, nonstrict ones by the symbol $\preceq$.

⌈7′4⌋ Cones

Let $(X, \prec)$ be a poset and let $a \in X$. The set $\{x \in X \mid a \prec x\}$ is the *upper cone* of a, and the set $\{x \in X \mid x \prec a\}$ the *lower cone* of a. The element a does not belong to its cones. Adding a to them, we obtain *completed* cones: the *upper completed cone* or *star* $C^+_X(a) = \{x \in X \mid a \preceq x\}$ and the *lower completed cone* $C^-_X(a) = \{x \in X \mid x \preceq a\}$.

7.I Properties of Cones. *Let $(X, \prec)$ be a poset. Then we have:*

(1) *$C^+_X(b) \subset C^+_X(a)$, provided that $b \in C^+_X(a)$;*

(2) *$a \in C^+_X(a)$ for each $a \in X$;*

(3) *$C^+_X(a) = C^+_X(b)$ implies $a = b$.*

7.J Cones Determine an Order. *Let X be an arbitrary set. Suppose for each $a \in X$ we fix a subset $C_a \subset X$ so that*

(1) *$b \in C_a$ implies $C_b \subset C_a$,*

(2) *$a \in C_a$ for each $a \in X$, and*

(3) *$C_a = C_b$ implies $a = b$.*

We write $a \prec b$ if $b \in C_a$. Then the relation $\prec$ is a nonstrict order on X, and for this order we have $C^+_X(a) = C_a$.

7.2. Let $C \subset \mathbb{R}^3$ be a set. Consider the relation $\lhd_C$ on $\mathbb{R}^3$ defined as follows: $a \lhd_C b$ if $b - a \in C$. What properties of C imply that $\lhd_C$ is a partial order on $\mathbb{R}^3$? What are the upper and lower cones in the poset $(\mathbb{R}^3, \lhd_C)$?

7.3. Prove that each convex cone C in $\mathbb{R}^3$ with vertex $(0, 0, 0)$ and such that $P \cap C = \{(0, 0, 0)\}$ for some plane P satisfies the conditions found in the solution to Problem *7.2*.

7.4. Consider the space-time $\mathbb{R}^4$ of special relativity theory, where points represent moment-point events and the first three coordinates x_1, x_2 and x_3 are the spatial coordinates, while the fourth one, t, is the time. This space carries a relation, "the *event* (x_1, x_2, x_3, t) *precedes* (and may influence) the *event* $(\widetilde{x}_1, \widetilde{x}_2, \widetilde{x}_3, \widetilde{t})$". The relation is defined by the inequality

$$c(\tilde{t} - t) \geq \sqrt{(\widetilde{x}_1 - x_1)^2 + (\widetilde{x}_2 - x_2)^2 + (\widetilde{x}_3 - x_3)^2}.$$

Is this a partial order? If yes, then what are the upper and lower cones of an event?

7.5. Answer the versions of questions of the preceding problem in the case of two- and three-dimensional analogs of this space, where the number of spatial coordinates is 1 and 2, respectively.

⌈7′5⌋ Position of an Element with Respect to a Set

Let $(X, \prec)$ be a poset, $A \subset X$ a subset. Then b is the *greatest element* of A if $b \in A$ and $c \preceq b$ for every $c \in A$. Similarly, b is the *smallest element* of A if $b \in A$ and $b \preceq c$ for every $c \in A$.

7.K. An element $b \in A$ is the smallest element of A iff $A \subset C_X^+(b)$; an element $b \in A$ is the greatest element of A iff $A \subset C_X^-(b)$.

7.L. Each set has at most one greatest and at most one smallest element.

An element b of a set A is a *maximal* element of A if A contains no element c such that $b \prec c$. An element b is a *minimal* element of A if A contains no element c such that $c \prec b$.

7.M. An element b of A is maximal iff $A \cap C_X^-(b) = b$; an element b of A is minimal iff $A \cap C_X^+(b) = b$.

7.6. Riddle. 1) How are the notions of maximal and greatest elements related? 2) What can you say about a poset in which these notions coincide for each subset?

⌈7′6⌋ Linear Orders

Please, notice: the definition of a strict order does not require that for any $a, b \in X$ we have either $a \prec b$, or $b \prec a$, or $a = b$. The latter condition is called a *trichotomy*. In terms of the corresponding nonstrict order, it is reformulated as follows: any two elements $a, b \in X$ are *comparable*: either $a \preceq b$, or $b \preceq a$.

A strict order satisfying trichotomy is *linear* (or *total*). The corresponding poset is *linearly* ordered (or *totally* ordered). It is also called just an *ordered set*.[9] Some orders do satisfy trichotomy.

7.N. The order $<$ on the set $\mathbb{R}$ of real numbers is linear.

This is the most important example of a linearly ordered set. The words and images rooted in it are often extended to all linearly ordered sets. For example, cones are called *rays*, upper cones become *right rays*, while lower cones become *left rays*.

7.7. A poset $(X, \prec)$ is linearly ordered iff $X = C_X^+(a) \cup C_X^-(a)$ for each $a \in X$.

7.8. The order $a \mid b$ on the set $\mathbb{N}$ of positive integers is not linear.

7.9. For which X is the relation of inclusion on the set of all subsets of X a linear order?

[9] Quite a bit of confusion was brought into the terminology by Bourbaki. At that time, linear orders were called orders, nonlinear orders were called partial orders, and, in occasions when it was not known if the order under consideration was linear, the fact that this was unknown was explicitly stated. Bourbaki suggested to drop the word *partial*. Their motivation for this was that a partial order is a phenomenon more general than a linear order, and hence deserves a shorter and simpler name. This suggestion was commonly accepted in the French literature, but in English literature it would imply abolishing a nice short word, *poset*, which seems to be an absolutely impossible thing to do.

⌈7′7⌋ Topologies Determined by Linear Order

7.O. *Let $(X, \prec)$ be a linearly ordered set. Then the set X itself and all right rays of X, i.e., sets of the form $\{x \in X \mid a \prec x\}$, where a runs through X, constitute a base for a topological structure in X.*

The topological structure determined by this base is the *right ray topology* of the linearly ordered set $(X, \prec)$. The *left ray topology* is defined similarly: it is generated by the base consisting of X and sets of the form $\{x \in X \mid x \prec a\}$ with $a \in X$.

7.10. The topology of the arrow (see Section 2) is the right ray topology of the half-line $[0, \infty)$ equipped with the order $<$.

7.11. Riddle. To what extent is the assumption that the order be linear necessary in Theorem *7.O*? Find a weaker condition that implies the conclusion of Theorem *7.O* and allows us to speak about the topological structure described in Problem *2.2* as the right ray topology of an appropriate partial order on the plane.

7.P. *Let $(X, \prec)$ be a linearly ordered set. Then the subsets of X having the forms*

- *$\{x \in X \mid a \prec x\}$, where a runs through X,*
- *$\{x \in X \mid x \prec a\}$, where a runs through X,*
- *$\{x \in X \mid a \prec x \prec b\}$, where a and b run through X*

constitute a base for a topological structure in X.

The topological structure determined by this base is the *interval topology* of the linearly ordered set $(X, \prec)$.

7.12. Prove that the interval topology is the smallest topological structure containing the right ray and left ray topological structures.

7.Q. *The canonical topology of the line is the interval topology of $(\mathbb{R}, <)$.*

⌈7′8⌋ Poset Topology

7.R. *Let $(X, \preceq)$ be a poset. Then the subsets of X having the form $\{x \in X \mid a \preceq x\}$, where a runs through the entire X, constitute a base for a topological structure in X.*

The topological structure generated by this base is the *poset topology*.

7.S. *In the poset topology, each point $a \in X$ has the smallest (with respect to inclusion) neighborhood. This is $\{x \in X \mid a \preceq x\}$.*

7.T. *The following properties of a topological space are equivalent:*

(1) *each point has a smallest neighborhood,*

(2) *the intersection of any collection of open sets is open,*

(3) *the union of any collection of closed sets is closed.*

A space satisfying the conditions of Theorem *7.T* is a ***smallest neighborhood space***.[10] In such a space, open and closed sets satisfy the same conditions. In particular, the set of all closed sets of a smallest neighborhood space is also a topological structure, which is ***dual*** to the original one. It corresponds to the opposite partial order.

7.13. How to characterize points open in the poset topology in terms of the partial order? Answer the same question about closed points. (Slightly abusing the terminology, here by *points* we mean the corresponding singletons.)

7.14. Directly describe open sets in the poset topology of $\mathbb{R}$ with order $<$.

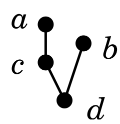
7.15. Consider a partial order on the set $\{a, b, c, d\}$ where the strict inequalities are: $c \prec a$, $d \prec c$, $d \prec a$, and $d \prec b$. Check that this is a partial order and the corresponding poset topology is the topology of described in Problem *2.3* (1).

7.16. Describe the closure of a point in a poset topology.

7.17. Which singletons are dense in a poset topology?

⌈7′9⌋ How to Draw a Poset

Now we can explain the pictogram , by which we denote the space introduced in Problem *2.3*(1). It describes the partial order on $\{a, b, c, d\}$ that determines the topology of this space by *7.15*. Indeed, if we place a, b, c, and d, i.e., the elements of the set under consideration, at vertices of the graph of the pictogram, as shown in the picture, then the vertices marked by comparable elements are connected by a segment or ascending broken line, and the greater element corresponds to the higher vertex.

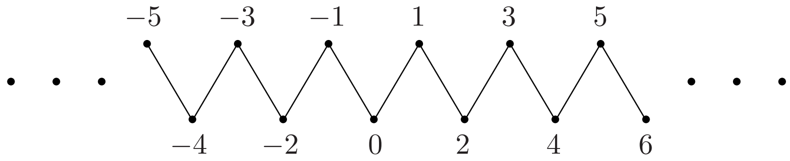


In this way, we can represent any finite poset by a diagram. Elements of the poset are represented by points. We have $a \prec b$ if and only if the following two conditions are fulfilled: 1) the point representing b lies above the point representing a, and 2) the two points are connected either by a segment or by a polyline consisting of segments that connect points representing intermediate elements of a chain $a \prec c_1 \prec c_2 \prec \dots \prec c_n \prec b$. We could have connected by a segment any two points corresponding to comparable elements, but this would make the diagram excessively cumbersome. This is why the segments that are determined by the others via transitivity are not drawn. Such a diagram representing a poset is its ***Hasse diagram***.

7.U. Prove that any finite poset is determined by a Hasse diagram.

[10]This class of topological spaces was introduced and studied by P. S. Alexandrov in 1935. Alexandrov called them ***discrete***. Nowadays, the term discrete space is used for a much narrower class of topological spaces (see Section 2). The term ***smallest neighborhood space*** was introduced by Christer Kiselman.

7.V. Describe the poset topology on the set $\mathbb{Z}$ of integers defined by the following Hasse diagram:



The space of Problem *7.V* is the ***digital line***, or ***Khalimsky line***. In this space, each even number is closed and each odd one is open.

7.18. Associate with each even integer $2k$ the interval $(2k-1, 2k+1)$ of length 2 centered at this point, and with each odd integer $2k-1$, the singleton $\{2k-1\}$. Prove that a set of integers is open in the Khalimsky topology iff the union of sets associated to its elements is open in $\mathbb{R}$ with the standard topology.

7.19. Among the topological spaces described in Section 2, find all those obtained as posets with the poset topology. In the cases of finite sets, draw Hasse diagrams describing the corresponding partial orders.

8. Cyclic Orders

⌈8′1⌋ Cyclic Orders in Finite Sets

Recall that a *cyclic order* on a finite set X is a linear order considered up to cyclic permutation. The linear order allows us to enumerate elements of the set X by positive integers, so that $X = \{x_1, x_2, \dots, x_n\}$. A cyclic permutation transposes the first k elements with the last $n-k$ elements without changing the order inside each of the two parts of the set:

$$(x_1, x_2, \dots, x_k, x_{k+1}, x_{k+2}, \dots, x_n) \mapsto (x_{k+1}, x_{k+2}, \dots, x_n, x_1, x_2, \dots, x_k).$$

When we consider a cyclic order, it makes no sense to say that one of its elements is greater than another one, since an appropriate cyclic permutation puts the two elements in the opposite order. However, it makes sense to say that an element *immediately* precedes another one. Certainly, the very last element immediately precedes the very first one: indeed, a nontrivial cyclic permutation puts the first element immediately after the last one.

In a cyclically ordered finite set, each element a has a unique element b next to a, i.e., which follows a immediately. This determines a map of the set onto itself, namely, the simplest cyclic permutation

$$x_i \mapsto \begin{cases} x_{i+1} & \text{if } i < n, \\ x_1 & \text{if } i = n. \end{cases}$$

This permutation acts transitively (i.e., any element is mapped to any other one by an appropriate iteration of the permutation).

8.A. *Any map $T : X \to X$ that transitively acts on X determines a cyclic order on X such that each $a \in X$ precedes $T(a)$.*

8.B. *An n-element set possesses exactly $(n-1)!$ pairwise distinct cyclic orders.*

In particular, a two-element set has only one cyclic order (which is so uninteresting that sometimes it is said to make no sense), while any three-element set possesses two cyclic orders.

⌈8′2x⌋ Cyclic Orders in Infinite Sets

One can consider cyclic orders on an infinite set. However, most of what was said above does not apply to cyclic orders on infinite sets without an adjustment. In particular, most of them cannot be described by showing pairs of elements that are next to each other. For example, points of a circle can be cyclically ordered clockwise (or counter-clockwise), but no point immediately follows another point with respect to this cyclic order.

Such “continuous” cyclic orders are defined almost in the same way as cyclic orders on finite sets were defined above. The difference is that sometimes it is impossible to define *cyclic permutations of a set* in the necessary quantity, and we have to replace them by *cyclic transformations of linear orders*. Namely, a cyclic order is defined as a linear order considered up to cyclic transformations, where by a *cyclic transformation* of a linear order $\prec$ on a set X we mean a passage from $\prec$ to a linear order $\prec'$ such that X splits into subsets A and B such that the restrictions of $\prec$ and $\prec'$ to each of them coincide, while $a \prec b$ and $b \prec' a$ for any $a \in A$ and $b \in B$.

8.Cx. *Existence of a cyclic transformation transforming linear orders to each other determines an equivalence relation on the set of all linear orders on a set.*

A *cyclic order* on a set is an equivalence class of linear orders with respect to the above equivalence relation.

8.Dx. Prove that for a finite set this definition is equivalent to the definition in the preceding section.

8.Ex. Prove that the cyclic “counter-clockwise” order on a circle can be defined along the definition of this section, but cannot be defined as a linear order modulo cyclic transformations of the set for whatever definition of cyclic transformations of circle. Describe the linear orders on the circle that determine this cyclic order up to cyclic transformations of orders.

8.Fx. Let A be a subset of a set X. If two linear orders $\prec'$ and $\prec$ on X are obtained from each other by a cyclic transformation, then their restrictions to A are also obtained from each other by a cyclic transformation.

8.Gx Corollary. A cyclic order on a set induces a well-defined cyclic order on every subset of this set.

8.Hx. *A cyclic order on a set X can be recovered from the cyclic orders induced by it on all three-element subsets of X.*

8.Hx.1. A cyclic order on a set X can be recovered from the cyclic orders induced by it on all three-element subsets of X containing a fixed element $a \in X$.

Theorem *8.Hx* allows us to describe a cyclic order as a ternary relation. Namely, let a, b, and c be elements of a cyclically ordered set. Then we write $[a \prec b \prec c]$ if the induced cyclic order on $\{a, b, c\}$ is determined by the linear order in which the inequalities in the brackets hold true (i.e., b follows a and c follows b).

8.Ix. *Let X be a cyclically ordered set. Then the ternary relation $[a \prec b \prec c]$ on X has the following properties:*

(1) *for any pairwise distinct* $a, b, c \in X$, *we have either* $[a \prec b \prec c]$, *or* $[b \prec a \prec c]$, *but not both;*

(2) $[a \prec b \prec c]$, *iff* $[b \prec c \prec a]$, *iff* $[c \prec a \prec b]$, *for any* $a, b, c \in X$;

(3) *if* $[a \prec b \prec c]$ *and* $[a \prec c \prec d]$, *then* $[a \prec b \prec d]$.

Vice versa, a ternary relation on X *having these four properties determines a cyclic order on the set* X.

⌈8′3x⌋ Topology of Cyclic Order

8.Jx. *Let* X *be a cyclically ordered set. Then the sets that belong to the interval topology of every linear order determining the cyclic order on* X *constitute a topological structure in* X.

The topology defined in *8.Jx* is the *cyclic order topology*.

8.Kx. The cyclic order topology determined by the cyclic counterclockwise order of S^1 is the topology generated by the metric $\rho(x, y) = |x - y|$ on $S^1 \subset \mathbb{C}$.

Chapter II

Continuity

9. Set-Theoretic Digression: Maps

⌈9′1⌋ Maps and Main Classes of Maps

A *map* f of a set X to a set Y is a triple consisting of X, Y, and a rule,[1] which assigns to every element of X exactly one element of Y. There are other words with the same meaning: *mapping*, *function*, etc. (Special kinds of maps may have special names like *functional*, *operator*, etc.)

If f is a map of X to Y, then we write $f : X \to Y$, or $X \xrightarrow{f} Y$. The element b of Y assigned by f to an element a of X is denoted by $f(a)$ and called the *image* of a under f, or the f-*image* of a. We write $b = f(a)$, or $a \overset{f}{\mapsto} b$, or $f : a \mapsto b$. We also define maps by formulas like $f : X \to Y : a \mapsto b$, where b is explicitly expressed in terms of a.

A map $f : X \to Y$ is a *surjective map*, or just a *surjection* if every element of Y is the image of at least one element of X. (We also say that f is *onto*.) A map $f : X \to Y$ is an *injective map*, *injection*, or *one-to-one map* if every element of Y is the image of at most one element of X. A map is a *bijective map*, *bijection*, or *invertible map* if it is both surjective and injective.

[1]Certainly, the rule (as everything in set theory) may be thought of as a set. Namely, we consider the set of the ordered pairs (x, y) with $x \in X$ and $y \in Y$ such that the rule assigns y to x. This is the *graph* of f. It is a subset of $X \times Y$. (Recall that $X \times Y$ is the set of all ordered pairs (x, y) with $x \in X$ and $y \in Y$.)

⌈9′2⌋ Image and Preimage

The *image* of a set $A \subset X$ under a map $f : X \to Y$ is the set of images of all points of A. It is denoted by $f(A)$. Thus, we have

$$f(A) = \{f(x) \mid x \in A\}.$$

The image of the entire set X (i.e., the set $f(X)$) is the *image* of f. It is denoted by $\operatorname{Im} f$.

The *preimage* of a set $B \subset Y$ under a map $f : X \to Y$ is the set of elements of X with images in B. It is denoted by $f^{-1}(B)$. Thus, we have

$$f^{-1}(B) = \{a \in X \mid f(a) \in B\}.$$

Be careful with these terms: their etymology can be misleading. For example, the image of the preimage of a set B can differ from B, and even if it does not differ, it may happen that the preimage is not the only set with this property. Hence, the preimage *cannot* be defined as a set whose image is the given set.

9.A. We have $f\bigl(f^{-1}(B)\bigr) \subset B$ for any map $f : X \to Y$ and any $B \subset Y$.

9.B. $f\bigl(f^{-1}(B)\bigr) = B$ iff $B \subset \operatorname{Im} f$.

9.C. Let $f : X \to Y$ be a map, and let $B \subset Y$ be such that $f\bigl(f^{-1}(B)\bigr) = B$. Then the following statements are equivalent:

(1) $f^{-1}(B)$ is the unique subset of X whose image equals B;

(2) for any $a_1, a_2 \in f^{-1}(B)$, the equality $f(a_1) = f(a_2)$ implies $a_1 = a_2$.

9.D. A map $f : X \to Y$ is an injection iff for each $B \subset Y$ such that $f\bigl(f^{-1}(B)\bigr) = B$ the preimage $f^{-1}(B)$ is the unique subset of X with image equal to B.

9.E. We have $f^{-1}\bigl(f(A)\bigr) \supset A$ for any map $f : X \to Y$ and any $A \subset X$.

9.F. $f^{-1}\bigl(f(A)\bigr) = A$ iff $f(A) \cap f(X \smallsetminus A) = \varnothing$.

9.1. Do the following equalities hold true for any $A, B \subset Y$ and $f : X \to Y$:

$$f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B), \tag{10}$$

$$f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B), \tag{11}$$

$$f^{-1}(Y \smallsetminus A) = X \smallsetminus f^{-1}(A)? \tag{12}$$

9.2. Do the following equalities hold true for any $A, B \subset X$ and $f : X \to Y$:

$$f(A \cup B) = f(A) \cup f(B), \tag{13}$$

$$f(A \cap B) = f(A) \cap f(B), \tag{14}$$

$$f(X \smallsetminus A) = Y \smallsetminus f(A)? \tag{15}$$

9.3. Give examples in which two of the above equalities (13)–(15) are false.

9.4. Replace false equalities of *9.2* by correct inclusions.

9.5. Riddle. What simple condition on $f : X \to Y$ should be imposed in order to make all equalities of *9.2* correct for any $A, B \subset X$?

9.6. Prove that for any map $f : X \to Y$ and any subsets $A \subset X$ and $B \subset Y$ we have:

$$B \cap f(A) = f\big(f^{-1}(B) \cap A\big).$$

⌈9′3⌋ Identity and Inclusion

The *identity map* of a set X is the map $\mathrm{id}_X : X \to X : x \mapsto x$. It is denoted just by id if there is no ambiguity. If A is a subset of X, then the map $\mathrm{in}_A : A \to X : \ x \mapsto x$ is the *inclusion map*, or just *inclusion*, of A into X. It is denoted just by in when A and X are clear.

9.G. The preimage of a set B under the inclusion $\mathrm{in} : A \to X$ is $B \cap A$.

⌈9′4⌋ Composition

The *composition* of maps $f : X \to Y$ and $g : Y \to Z$ is the map $g \circ f : X \to Z : x \mapsto g\big(f(x)\big)$.

9.H Associativity of Composition. We have $h \circ (g \circ f) = (h \circ g) \circ f$ for any maps $f : X \to Y$, $g : Y \to Z$, and $h : Z \to U$.

9.I. We have $f \circ \mathrm{id}_X = f = \mathrm{id}_Y \circ f$ for any $f : X \to Y$.

9.J. A composition of injections is injective.

9.K. If the composition $g \circ f$ is injective, then so is f.

9.L. A composition of surjections is surjective.

9.M. If the composition $g \circ f$ is surjective, then so is g.

9.N. A composition of bijections is a bijection.

9.7. Let a composition $g \circ f$ be bijective. Is then f or g necessarily bijective?

⌈9′5⌋ Inverse and Invertible

A map $g : Y \to X$ is *inverse* to a map $f : X \to Y$ if $g \circ f = \mathrm{id}_X$ and $f \circ g = \mathrm{id}_Y$. A map having an inverse map is *invertible*.

9.O. A map is invertible iff it is a bijection.

9.P. If an inverse map exists, then it is unique.

⌈9′6⌋ Submaps

If $A \subset X$ and $B \subset Y$, then for every $f : X \to Y$ such that $f(A) \subset B$ we have a map $\mathrm{ab}(f) : A \to B : x \mapsto f(x)$, which is called the ***abbreviation*** of f to A and B, a ***submap***, or a ***submapping***. If $B = Y$, then $\mathrm{ab}(f) : A \to Y$ is denoted by $f|_A$ and called the ***restriction*** of f to A. If $B \neq Y$, then $\mathrm{ab}(f) : A \to B$ is denoted by $f|_{A,B}$ or even simply $f|$.

9.Q. The restriction of a map $f : X \to Y$ to $A \subset X$ is the composition of the inclusion $\mathrm{in} : A \to X$ and f. In other words, $f|_A = f \circ \mathrm{in}$.

9.R. Any submap (in particular, any restriction) of an injection is injective.

9.S. If a map possesses a surjective restriction, then it is surjective.

10. Continuous Maps

⌈10′1⌋ Definition and Main Properties of Continuous Maps

Let X and Y be two topological spaces. A map $f : X \to Y$ is *continuous* if the preimage of each open subset of Y is an open subset of X.

10.A. *A map is continuous iff the preimage of each closed set is closed.*

10.B. *The identity map of any topological space is continuous.*

10.C. Any constant map (i.e., a map with one-point image) is continuous.

10.1. Let Ω_1 and Ω_2 be two topological structures in a space X. Prove that the identity map

$$\mathrm{id} : (X, \Omega_1) \to (X, \Omega_2)$$

is continuous iff $\Omega_2 \subset \Omega_1$.

10.2. Let $f : X \to Y$ be a continuous map. Find out whether or not it is continuous with respect to

(1) a finer topology on X and the same topology on Y,
(2) a coarser topology on X and the same topology on Y,
(3) a finer topology on Y and the same topology on X,
(4) a coarser topology on Y and the same topology on X.

10.3. Let X be a discrete space, Y an arbitrary space. 1) Which maps $X \to Y$ are continuous? 2) Which maps $Y \to X$ are continuous for each topology on Y?

10.4. Let X be an indiscrete space, Y an arbitrary space. 2) Which maps $Y \to X$ are continuous? 1) Which maps $X \to Y$ are continuous for each topology on Y?

10.D. *Let A be a subspace of X. The inclusion* $\mathrm{in} : A \to X$ *is continuous.*

10.E. The topology Ω_A induced on $A \subset X$ by the topology of X is the coarsest topology on A with respect to which the inclusion $\mathrm{in} : A \to X$ is continuous.

10.5. Riddle. The statement *10.E* admits a natural generalization with the inclusion map replaced by an arbitrary map $f : A \to X$ of an arbitrary set A. Find this generalization.

10.F. *A composition of continuous maps is continuous.*

10.G. *A submap of a continuous map is continuous.*

10.H. *A map $f : X \to Y$ is continuous iff* $\mathrm{ab}(f) : X \to f(X)$ *is continuous.*

⌈10′2⌋ Reformulations of Definition

10.6. Prove that a map $f : X \to Y$ is continuous iff

$$\operatorname{Cl} f^{-1}(A) \subset f^{-1}(\operatorname{Cl} A)$$

for each $A \subset Y$.

10.7. Formulate and prove similar criteria of continuity in terms of $\operatorname{Int} f^{-1}(A)$ and $f^{-1}(\operatorname{Int} A)$. Do the same for $\operatorname{Cl} f(A)$ and $f(\operatorname{Cl} A)$.

10.8. Let Σ be a base for the topology on Y. Prove that a map $f : X \to Y$ is continuous iff $f^{-1}(U)$ is open for each $U \in \Sigma$.

⌈10′3⌋ More Examples

10.9. Consider the map

$$f : [0,2] \to [0,2] : f(x) = \begin{cases} x & \text{if } x \in [0,1), \\ 3-x & \text{if } x \in [1,2]. \end{cases}$$

Is it continuous (with respect to the topology induced from the real line)?

10.10. Consider the map f from the segment $[0,2]$ (with the relative topology induced by the topology of the real line) into the arrow (see Section 2) defined by the formula

$$f(x) = \begin{cases} x & \text{if } x \in [0,1], \\ x+1 & \text{if } x \in (1,2]. \end{cases}$$

Is it continuous?

10.11. Give an explicit characterization of continuous maps of $\mathbb{R}_{T_1}$ (see Section 2) to $\mathbb{R}$.

10.12. Which maps $\mathbb{R}_{T_1} \to \mathbb{R}_{T_1}$ are continuous?

10.13. Give an explicit characterization of continuous maps of the arrow to itself.

10.14. Let f be a map of the set $\mathbb{Z}_+$ of nonnegative numbers to $\mathbb{R}$ defined by the formula

$$f(x) = \begin{cases} 1/x & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Let $g : \mathbb{Z}_+ \to f(\mathbb{Z}_+)$ be the submap of f. Induce a topology on $\mathbb{Z}_+$ and $f(\mathbb{Z}_+)$ from $\mathbb{R}$. Are f and the map g^{-1} inverse to g continuous?

⌈10′4⌋ Behavior of Dense Sets Under Continuous Maps

10.15. Prove that the image of an everywhere dense set under a surjective continuous map is everywhere dense.

10.16. Is it true that the image of a nowhere dense set under a continuous map is nowhere dense?

10.17*. Do there exist a nowhere dense subset A of $[0,1]$ (with the topology induced from the real line) and a continuous map $f : [0,1] \to [0,1]$ such that $f(A) = [0,1]$?

⌈10′5⌋ Local Continuity

A map f from a topological space X to a topological space Y is *continuous at a point* $a \in X$ if for every neighborhood V of $f(a)$ the point a has a neighborhood U such that $f(U) \subset V$.

10.I. *A map $f : X \to Y$ is continuous iff it is continuous at each point of X.*

10.J. *Let X and Y be two metric spaces. A map $f : X \to Y$ is continuous at a point $a \in X$ iff each ball centered at $f(a)$ contains the image of a ball centered at a.*

10.K. *Let X and Y be two metric spaces. A map $f : X \to Y$ is continuous at a point $a \in X$ iff for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every point $x \in X$ the inequality $\rho(x, a) < \delta$ implies $\rho\big(f(x), f(a)\big) < \varepsilon$.*

Theorem *10.K* means that the definition of continuity usually studied in Calculus, when applicable, is equivalent to the above definition stated in terms of topological structures.

⌈10′6⌋ Properties of Continuous Functions

10.18. Let $f, g : X \to \mathbb{R}$ be two continuous functions. Prove that the functions $X \to \mathbb{R}$ defined by the formulas

$$x \mapsto f(x) + g(x), \tag{16}$$
$$x \mapsto f(x)g(x), \tag{17}$$
$$x \mapsto f(x) - g(x), \tag{18}$$
$$x \mapsto \big|f(x)\big|, \tag{19}$$
$$x \mapsto \max\{f(x), g(x)\}, \tag{20}$$
$$x \mapsto \min\{f(x), g(x)\} \tag{21}$$

are continuous.

10.19. Prove that if $0 \notin g(X)$, then the function

$$X \to \mathbb{R} : x \mapsto \frac{f(x)}{g(x)}$$

is also continuous.

10.20. Find a sequence of continuous functions $f_i : \mathbb{R} \to \mathbb{R}$, $(i \in \mathbb{N})$, such that the function

$$\mathbb{R} \to \mathbb{R} : x \mapsto \sup\{\, f_i(x) \mid i \in \mathbb{N}\,\}$$

is not continuous.

10.21. Let X be a topological space. Prove that a function $f : X \to \mathbb{R}^n : x \mapsto (f_1(x), \dots, f_n(x))$ is continuous iff so are all functions $f_i : X \to \mathbb{R}$ with $i = 1, \dots, n$.

Real $p \times q$ matrices form a space $Mat(p \times q, \mathbb{R})$, which differs from $\mathbb{R}^{pq}$ only in the way its natural coordinates are numbered (they are numbered by pairs of indices).

10.22. Let $f : X \to Mat(p \times q, \mathbb{R})$ and $g : X \to Mat(q \times r, \mathbb{R})$ be two continuous maps. Prove that the map

$$X \to Mat(p \times r, \mathbb{R}) : x \mapsto g(x)f(x)$$

is also continuous.

Recall that $GL(n; \mathbb{R})$ is the subspace of $Mat(n \times n, \mathbb{R})$ consisting of all invertible matrices.

10.23. Let $f : X \to GL(n; \mathbb{R})$ be a continuous map. Prove that $X \to GL(n; \mathbb{R}) : x \mapsto (f(x))^{-1}$ is also continuous.

⌈10′7⌋ Continuity of Distances

10.L. *For every subset A of a metric space X, the function $X \to \mathbb{R} : x \mapsto \rho(x, A)$ (see Section 4) is continuous.*

10.24. Prove that the metric topology of a metric space X is the coarsest topology with respect to which the function $X \to \mathbb{R} : x \mapsto \rho(x, A)$ is continuous for every $A \subset X$.

⌈10′8⌋ Isometry

A map f of a metric space X to a metric space Y is an *isometric embedding* if $\rho\big(f(a), f(b)\big) = \rho(a, b)$ for any $a, b \in X$. A bijective isometric embedding is an *isometry*.

10.M. Every isometric embedding is injective.

10.N. Every isometric embedding is continuous.

⌈10′9⌋ Contractive Maps

A map $f : X \to X$ of a metric space X is *contractive* if there exists $\alpha \in (0, 1)$ such that $\rho\big(f(a), f(b)\big) \le \alpha\rho(a, b)$ for any $a, b \in X$.

10.25. Prove that every contractive map is continuous.

Let X and Y be two metric spaces. A map $f : X \to Y$ is a *Hölder* map if there exist $C > 0$ and $\alpha > 0$ such that $\rho\big(f(a), f(b)\big) \le C\rho(a, b)^\alpha$ for any $a, b \in X$.

10.26. Prove that every Hölder map is continuous.

⌈10′10⌋ Sets Defined by Systems of Equations and Inequalities

10.O. *Let $f_1, \dots, f_n : X \to \mathbb{R}$ be continuous functions. Then the subset of X formed by solutions to the system of equations*

$$f_1(x) = \cdots = f_n(x) = 0$$

is closed.

10.P. *Let $f_1, \dots, f_n : X \to \mathbb{R}$ be continuous functions. Then the subset of X formed by solutions to the system of inequalities*

$$f_1(x) \geq 0, \dots, f_n(x) \geq 0$$

is closed, while the set of solutions to the system of inequalities

$$f_1(x) > 0, \dots, f_n(x) > 0$$

is open.

10.27. Where in *10.O* and *10.P* can a finite system be replaced by an infinite one?

10.28. Prove that in $\mathbb{R}^n$ $(n \geq 1)$ every proper algebraic set (i.e., a set defined by algebraic equations) is nowhere dense.

⌈10′11⌋ Set-Theoretic Digression: Covers

A collection Γ of subsets of a set X is a *cover* or a *covering* of X if X is the union of sets in Γ, i.e., $X = \bigcup_{A \in \Gamma} A$. In this case, elements of Γ *cover* X.

These words also have a more general meaning. A collection Γ of subsets of a set Y is a *cover* or a *covering* of a set $X \subset Y$ if X is contained in the union of the sets in Γ, i.e., $X \subset \bigcup_{A \in \Gamma} A$. In this case, the sets in Γ are also said to *cover* X.

⌈10′12⌋ Fundamental Covers

Consider a cover Γ of a topological space X. Each element of Γ inherits a topological structure from X. When do these structures uniquely determine the topology of X? In particular, what conditions on Γ ensure that the continuity of a map $f : X \to Y$ follows from the continuity of its restrictions to elements of Γ? To answer these questions, solve Problems *10.29–10.30* and *10.Q–10.V*.

10.29. Find out whether or not this is true for the following covers:

(1) $X = [0, 2]$, and $\Gamma = \{[0, 1], (1, 2]\}$;
(2) $X = [0, 2]$, and $\Gamma = \{[0, 1], [1, 2]\}$;
(3) $X = \mathbb{R}$, and $\Gamma = \{\mathbb{Q}, \mathbb{R} \setminus \mathbb{Q}\}$;
(4) $X = \mathbb{R}$, and Γ is the set of all one-point subsets of $\mathbb{R}$.

A cover Γ of a space X is *fundamental* if: a set $U \subset X$ is open iff for every $A \in \Gamma$ the set $U \cap A$ is open in A.

10.Q. A cover Γ of a space X is fundamental iff: a set $U \subset X$ is open, provided $U \cap A$ is open in A for every $A \in \Gamma$.

10.R. A cover Γ of a space X is fundamental iff: a set $F \subset X$ is closed, provided that $F \cap A$ is closed in A for every $A \in \Gamma$.

10.30. The cover of a topological space by singletons is fundamental iff the space is discrete.

A cover of a topological space is *open* (respectively, *closed*) if it consists of open (respectively, closed) sets. A cover of a topological space is *locally finite* if every point of the space has a neighborhood meeting only a finite number of elements of the cover.

10.S. *Every open cover is fundamental.*

10.T. *A finite closed cover is fundamental.*

10.U. Every locally finite closed cover is fundamental.

10.V. *Let Γ be a fundamental cover of a topological space X, and let $f : X \to Y$ be a map. If the restriction of f to each element of Γ is continuous, then so is f.*

A cover Γ' is a *refinement* of a cover Γ if every element of Γ' is contained in an element of Γ.

10.31. Prove that if a cover Γ' is a refinement of a cover Γ and Γ' is fundamental, then so is Γ.

10.32. Let Δ be a fundamental cover of a topological space X, and let Γ be a cover of X such that $\Gamma_A = \{U \cap A \mid U \in \Gamma\}$ is a fundamental cover for the subspace $A \subset X$ for every $A \in \Delta$. Prove that Γ is a fundamental cover of X.

10.33. Prove that the property of being fundamental is local, i.e., if every point of a space X has a neighborhood V such that $\Gamma_V = \{U \cap V \mid U \in \Gamma\}$ is a fundamental cover of V, then Γ is fundamental.

⌈10′13x⌋ Monotone Maps

Let $(X, \prec)$ and $(Y, \prec)$ be two posets. A map $f : X \to Y$ is

- (*non-strictly*) *monotonically increasing* or just *monotone* if $f(a) \preceq f(b)$ for any $a, b \in X$ with $a \preceq b$;
- (*non-strictly*) *monotonically decreasing* or *antimonotone* if $f(b) \preceq f(a)$ for any $a, b \in X$ with $a \preceq b$;
- *strictly monotonically increasing* or just *strictly monotone* if $f(a) \prec f(b)$ for any $a, b \in X$ with $a \prec b$;
- *strictly monotonically decreasing* or *strictly antimonotone* if $f(b) \prec f(a)$ for any $a, b \in X$ with $a \prec b$.

10.Wx. *Let X and Y be two linearly ordered sets. Then any surjective strictly monotone or antimonotone map $X \to Y$ is continuous with respect to the interval topology on X and Y.*

***10.34*x.** Show that the surjectivity condition in *10.Wx* is needed.

***10.35*x.** Is it possible to remove the word *strictly* from the hypothesis of Theorem *10.Wx*?

***10.36*x.** In the assumptions of Theorem *10.Wx*, is f continuous with respect to the right-ray or left-ray topologies?

***10.X*x.** *A map $f : X \to Y$ of a poset to a poset is monotone increasing iff it is continuous with respect to the poset topologies on X and Y.*

⌈10′14x⌋ Gromov–Hausdorff Distance

***10.37*x.** For any metric spaces X and Y, there exists a metric space Z such that X and Y can be isometrically embedded in Z.

Isometrically embedding two metric space in a single one, we can consider the Hausdorff distance between their images (see Section 4′15x). The infimum of such Hausdorff distances over all pairs of isometric embeddings of metric spaces X and Y in metric spaces is the ***Gromov–Hausdorff distance*** between X and Y.

***10.38*x.** Do there exist metric spaces with infinite Gromov–Hausdorff distance?

***10.39*x.** Prove that the Gromov–Hausdorff distance is symmetric and satisfies the triangle inequality.

10.40*x. *Riddle. In what sense can the Gromov–Hausdorff distance satisfy the first axiom of metric?

⌈10′15x⌋ Functions on the Cantor Set and Square-Filling Curves

Recall that the Cantor set K is the set of real numbers that are presented as sums of series of the form $\sum_{n=1}^{\infty} a_n/3^n$ with $a_n \in \{0, 2\}$.

***10.41*x.** Consider the map

$$\gamma_1 : K \to [0,1] : \sum_{n=1}^{\infty} \frac{a_n}{3^n} \mapsto \frac{1}{2}\sum_{n=1}^{\infty} \frac{a_n}{2^n}.$$

Prove that γ_1 is a continuous surjection. Sketch the graph of γ_1.

***10.42*x.** Prove that the function

$$K \to K : \sum_{n=1}^{\infty} \frac{a_n}{3^n} \mapsto \sum_{n=1}^{\infty} \frac{a_{2n}}{3^n}$$

is continuous.

Denote by K^2 the set $\{(x, y) \in \mathbb{R}^2 \mid x \in K,\ y \in K\}$.

***10.43*x.** Prove that the map

$$\gamma_2 : K \to K^2 : \sum_{n=1}^{\infty} \frac{a_n}{3^n} \mapsto \left(\sum_{n=1}^{\infty} \frac{a_{2n-1}}{3^n}, \sum_{n=1}^{\infty} \frac{a_{2n}}{3^n} \right)$$

is a continuous surjection.

The unit segment $[0, 1]$ is denoted by I, while the set

$$\{(x_1, \dots, x_n) \subset \mathbb{R}^n \mid 0 \le x_i \le 1 \text{ for each } i\}$$

is denoted by I^n and called the (unit) *n-cube*.

10.44x. Prove that the map $\gamma_3 : K \to I^2$ defined as the composition of $\gamma_2 : K \to K^2$ and $K^2 \to I^2 : (x, y) \mapsto (\gamma_1(x), \gamma_1(y))$ is a continuous surjection.

10.45x. Prove that the map $\gamma_3 : K \to I^2$ is a restriction of a continuous map. (Cf. *2.Jx.2*.)

The latter map is a continuous surjection $I \to I^2$. Thus, this is a curve *filling* the square. A curve with this property was first constructed by G. Peano in 1890. Though the construction sketched above involves the same ideas as Peano's original construction, the two constructions are slightly different. A lot of other similar examples have been found since then. You may find a nice survey of them in Hans Sagan's book *Space-Filling Curves*, Springer-Verlag 1994. Here is a sketch of Hilbert's construction.

10.46x. Prove that there exists a sequence of polygonal maps $f_n : I \to I^2$ such that

(1) f_n connects all centers of the 4^n equal squares with side $1/2^n$ forming an obvious subdivision of I^2;
(2) we have $\operatorname{dist}(f_n(x), f_{n-1}(x)) \le \sqrt{2}/2^{n+1}$ for any $x \in I$ (here, dist denotes the metric induced on I^2 by the standard Euclidean metric of $\mathbb{R}^2$).

10.47x. Prove that any sequence of paths $f_n : I \to I^2$ satisfying the conditions of *10.46x* converges to a map $f : I \to I^2$ (i.e., for any $x \in I$ there exists a limit $f(x) = \lim_{n\to\infty} f_n(x)$), this map is continuous, and its image $f(I)$ is dense in I^2.

10.48x.[2] Prove that any continuous map $I \to I^2$ with dense image is surjective.

10.49x. Generalize *10.43x - 10.48x* to obtain a continuous surjection of I onto I^n.

[2]Although this problem can be solved by using theorems that are well known from Calculus, we have to mention that it would be more appropriate to solve it after Section 17. Cf. Problems *17.P*, *17.U*, and *17.K*.

11. Homeomorphisms

⌈11′1⌋ Definition and Main Properties of Homeomorphisms

An invertible map $f : X \to Y$ is a *homeomorphism* if both this map and its inverse are continuous.

11.A. Find an example of a continuous bijection which is not a homeomorphism.

11.B. Find a continuous bijection $[0, 1) \to S^1$ which is not a homeomorphism.

11.C. The identity map of a topological space is a homeomorphism.

11.D. *A composition of homeomorphisms is a homeomorphism.*

11.E. *The inverse of a homeomorphism is a homeomorphism.*

⌈11′2⌋ Homeomorphic Spaces

A topological space X is *homeomorphic* to a space Y if there exists a homeomorphism $X \to Y$.

11.F. *Being homeomorphic is an equivalence relation.*

11.1. Riddle. How is Theorem *11.F* related to *11.C–11.E*?

⌈11′3⌋ Role of Homeomorphisms

11.G. Let $f : X \to Y$ be a homeomorphism. Then $U \subset X$ is open (in X) iff $f(U)$ is open (in Y).

11.H. A map $f : X \to Y$ is a homeomorphism iff f is a bijection and determines a bijection between the topological structures of X and Y.

11.I. Let $f : X \to Y$ be a homeomorphism. Then for every $A \subset X$

(1) A is closed in X iff $f(A)$ is closed in Y;
(2) $f(\operatorname{Cl} A) = \operatorname{Cl}(f(A))$;
(3) $f(\operatorname{Int} A) = \operatorname{Int}(f(A))$;
(4) $f(\operatorname{Fr} A) = \operatorname{Fr}(f(A))$;
(5) A is a neighborhood of a point $x \in X$ iff $f(A)$ is a neighborhood of the point $f(x)$;
(6) etc.

Therefore, homeomorphic spaces are completely identical from the topological point of view: a homeomorphism $X \to Y$ establishes a one-to-one correspondence between all phenomena in X and Y that can be expressed in terms of topological structures.[3]

$\lceil 11'4 \rfloor$ More Examples of Homeomorphisms

11.J. Let $f : X \to Y$ be a homeomorphism. Prove that for every $A \subset X$ the submap $\mathrm{ab}(f) : A \to f(A)$ is also a homeomorphism.

11.K. Prove that every isometry (see Section 10) is a homeomorphism.

11.L. Prove that every nondegenerate affine transformation of $\mathbb{R}^n$ is a homeomorphism.

11.M. *Let X and Y be two linearly ordered sets. Any strictly monotone surjection $f : X \to Y$ is a homeomorphism with respect to the interval topological structures in X and Y.*

11.N Corollary. *Any strictly monotone surjection $f : [a, b] \to [c, d]$ is a homeomorphism.*

11.2. Let R be a positive real. Prove that the *inversion*

$$\tau : \mathbb{R}^n \smallsetminus 0 \to \mathbb{R}^n \smallsetminus 0 : x \mapsto \frac{Rx}{|x|^2}$$

is a homeomorphism.

11.3. Let $\mathcal{H} = \{z \in \mathbb{C} \mid \operatorname{Im} z > 0\}$ be the upper half-plane, let $a, b, c, d \in \mathbb{R}$, and let $\begin{vmatrix} a & b \\ c & d \end{vmatrix} > 0$. Prove that

$$f : \mathcal{H} \to \mathcal{H} : z \mapsto \frac{az + b}{cz + d}$$

is a homeomorphism.

11.4. Let $f : \mathbb{R} \to \mathbb{R}$ be a bijection. Prove that f is a homeomorphism iff f is a monotone function.

11.5. 1) Prove that every bijection of an indiscrete space onto itself is a homeomorphism. Prove the same 2) for a discrete space and 3) $\mathbb{R}_{T_1}$.

11.6. Find all homeomorphisms of the space  (see Section 2) to itself.

11.7. Prove that every continuous bijection of the arrow onto itself is a homeomorphism.

[3]This phenomenon was used as a basis for defining the subject of topology in the first stages of its development, when the notion of topological space had not yet been developed. At that time, mathematicians studied only subspaces of Euclidean spaces, their continuous maps, and homeomorphisms. Felix Klein, in his famous Erlangen Program, classified various geometries that had emerged up to that time, like Euclidean, Lobachevsky, affine, and projective geometries, and defined topology as a part of geometry that deals with properties preserved by homeomorphisms. In fact, it was not assumed to be a program in the sense of something being planned, although it became a kind of program. It was a sort of dissertation presented by Klein for receiving a professor position at the Erlangen University.

11.8. Find two homeomorphic spaces X and Y and a continuous bijection $X \to Y$ which is not a homeomorphism.

11.9. Is $\gamma_2 : K \to K^2$ considered in Problem *10.43x* a homeomorphism? Recall that K is the Cantor set, $K^2 = \{(x, y) \in \mathbb{R}^2 \mid x \in K, y \in K\}$, and γ_2 is defined by

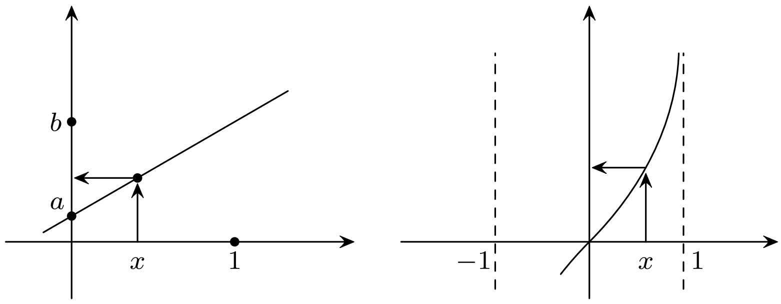
$$\sum_{k=1}^{\infty} \frac{a_k}{3^k} \mapsto \left(\sum_{k=1}^{\infty} \frac{a_{2k-1}}{3^k}, \sum_{k=1}^{\infty} \frac{a_{2k}}{3^k} \right).$$

⌈11′5⌋ Examples of Homeomorphic Spaces

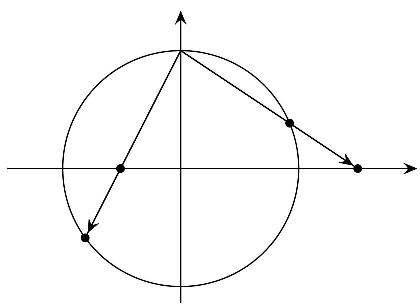
Below the homeomorphism relation is denoted by $\cong$. This notation is not commonly accepted. In other textbooks, you may see any sign close to, but distinct from $=$, e.g., $\sim$, $\simeq$, $\approx$, etc.

11.O. Prove that

(1) $[0, 1] \cong [a, b]$ for any $a < b$;
(2) $[0, 1) \cong [a, b) \cong (0, 1] \cong (a, b]$ for any $a < b$;
(3) $(0, 1) \cong (a, b)$ for any $a < b$;
(4) $(-1, 1) \cong \mathbb{R}$;
(5) $[0, 1) \cong [0, +\infty)$ and $(0, 1) \cong (0, +\infty)$.



11.P. Let $N = (0, 1) \in S^1$ be the North Pole of the unit circle. Prove that $S^1 \smallsetminus N \cong \mathbb{R}^1$.



11.Q. The graph of a continuous real-valued function defined on an interval is homeomorphic to the interval.

11.R. $S^n \smallsetminus \text{point} \cong \mathbb{R}^n$. (The first space is the "punctured sphere".)

Here, and sometimes below, our notation is slightly incorrect: in the curly brackets, we drop the initial part "$(x, y) \in \mathbb{R}^2 \mid$".

11.10. Prove that the following plane domains are homeomorphic.

(1) The whole plane $\mathbb{R}^2$;
(2) open square $\operatorname{Int} I^2 = \{ x, y \in (0, 1) \}$;
(3) open strip $\{ x \in (0, 1) \}$;
(4) open upper half-plane $\mathcal{H} = \{ y > 0 \}$;
(5) open half-strip $\{ x > 0,\ y \in (0, 1) \}$;
(6) open disk $B^2 = \{ x^2 + y^2 < 1 \}$;
(7) open rectangle $\{ a < x < b,\ c < y < d \}$;
(8) open quadrant $\{ x, y > 0 \}$;
(9) open angle $\{ x > y > 0 \}$;
(10) $\{ y^2 + |x| > x \}$, i.e., the plane without the ray $\{ y = 0 \le x \}$;
(11) open half-disk $\{ x^2 + y^2 < 1,\ y > 0 \}$;
(12) open sector $\{ x^2 + y^2 < 1,\ x > y > 0 \}$.

11.S. Prove that

(1) the closed disk D^2 is homeomorphic to the square $I^2 = \{ (x, y) \in \mathbb{R}^2 \mid x, y \in [0, 1] \}$;
(2) the open disk $B^2 = \{ (x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1 \}$ is homeomorphic to the open square $\operatorname{Int} I^2 = \{ (x, y) \in \mathbb{R}^2 \mid x, y \in (0, 1) \}$;
(3) the circle S^1 is homeomorphic to the boundary $\partial I^2 = I^2 \smallsetminus \operatorname{Int} I^2$ of the square.

11.T. Let $\Delta \subset \mathbb{R}^2$ be a planar bounded closed convex set with nonempty interior U. Prove that

(1) Δ is homeomorphic to the closed disk D^2;
(2) U is homeomorphic to the open disk B^2;
(3) $\operatorname{Fr} \Delta = \operatorname{Fr} U$ is homeomorphic to S^1.

11.11. In which of the assertions in *11.T* can we omit the assumption that the closed convex set Δ is bounded?

11.12. Classify up to homeomorphism all (nonempty) closed convex sets in the plane. (Make a list without repeats; prove that every such set is homeomorphic to a set in the list; postpone the proof of nonexistence of homeomorphisms till Section 12.)

11.13*. Generalize the previous three problems to the case of sets in $\mathbb{R}^n$ with arbitrary n.

The latter four problems show that angles are not essential in topology, i.e., for a line or the boundary of a domain the property of having angles is not preserved by homeomorphism. Here are two more problems in this direction.

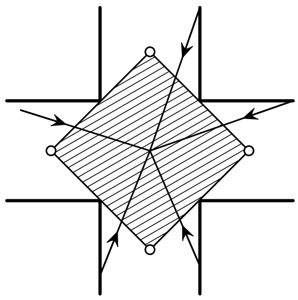
11.14. Prove that every simple (i.e., without self-intersections) closed polygon in $\mathbb{R}^2$ (as well as in $\mathbb{R}^n$ with $n > 2$) is homeomorphic to the circle S^1.

11.15. Prove that every nonclosed simple finite unit polyline in $\mathbb{R}^2$ (as well as in $\mathbb{R}^n$ with $n > 2$) is homeomorphic to the segment $[0, 1]$.

The following problem generalizes the technique used in the previous two problems and is actually used more often than it may seem at first glance.

11.16. Let X and Y be two topological spaces equipped with fundamental covers: $X = \bigcup_\alpha X_\alpha$ and $Y = \bigcup_\alpha Y_\alpha$. Suppose that $f : X \to Y$ is a map such that $f(X_\alpha) = Y_\alpha$ for each α and the submap $\mathrm{ab}(f) : X_\alpha \to Y_\alpha$ is a homeomorphism. Then f is a homeomorphism.

11.17. Prove that $\mathbb{R}^2 \smallsetminus \{\, |x|, |y| > 1 \,\} \cong I^2 \smallsetminus \{x, y \in \{0, 1\}\}$. (An "infinite cross" is homeomorphic to a square without vertices.)



11.18*. A nonempty set $\Sigma \subset \mathbb{R}^2$ is "star-shaped with respect to a point c" if Σ is a union of segments (and rays) with an endpoint at c. Prove that if Σ is open, then $\Sigma \cong B^2$. (What can you say about a closed star-shaped set with nonempty interior?)

11.19. Prove that the following plane figures are homeomorphic to each other. (See *11.10* for our agreement about notation.)

(1) A half-plane: $\{\, x \geq 0 \,\}$;
(2) a quadrant: $\{\, x, y \geq 0 \,\}$;
(3) an angle: $\{\, x \geq y \geq 0 \,\}$;
(4) a semi-open strip: $\{\, y \in [0, 1) \,\}$;
(5) a square without three sides: $\{\, 0 < x < 1,\ 0 \leq y < 1 \,\}$;
(6) a square without two sides: $\{\, 0 \leq x,\ y < 1 \,\}$;
(7) a square without a side: $\{\, 0 \leq x \leq 1,\ 0 \leq y < 1 \,\}$;
(8) a square without a vertex: $\{\, 0 \leq x,\ y \leq 1 \,\} \smallsetminus (1, 1)$;
(9) a disk without a boundary point: $\{\, x^2 + y^2 \leq 1,\ y \neq 1 \,\}$;
(10) a half-disk without the diameter: $\{\, x^2 + y^2 \leq 1,\ y > 0 \,\}$;
(11) a disk without a radius: $\{\, x^2 + y^2 \leq 1 \,\} \smallsetminus [0, 1]$;
(12) a square without a half of the diagonal: $\{\, |x| + |y| \leq 1 \,\} \smallsetminus [0, 1]$.

11.20. Prove that the following plane domains are homeomorphic to each other:

(1) punctured plane $\mathbb{R}^2 \smallsetminus (0, 0)$;
(2) punctured open disk $B^2 \smallsetminus (0, 0) = \{\, 0 < x^2 + y^2 < 1 \,\}$;
(3) annulus $\{\, a < x^2 + y^2 < b \,\}$, where $0 < a < b$;
(4) plane without a disk: $\mathbb{R}^2 \smallsetminus D^2$;
(5) plane without a square: $\mathbb{R}^2 \smallsetminus I^2$;

(6) plane without a segment: $\mathbb{R}^2 \smallsetminus [0,1]$;
(7) $\mathbb{R}^2 \smallsetminus \Delta$, where Δ is a closed bounded convex set with $\operatorname{Int}\Delta \neq \varnothing$.

11.21. Let $X \subset \mathbb{R}^2$ be the union of several segments with a common endpoint. Prove that the complement $\mathbb{R}^2 \smallsetminus X$ is homeomorphic to the punctured plane.

11.22. Let $X \subset \mathbb{R}^2$ be a simple nonclosed finite polyline. Prove that its complement $\mathbb{R}^2 \smallsetminus X$ is homeomorphic to the punctured plane.

11.23. Let $K = \{a_1, \dots, a_n\} \subset \mathbb{R}^2$ be a finite set. The complement $\mathbb{R}^2 \smallsetminus K$ is a ***plane with*** n ***punctures***. Prove that any two planes with n punctures are homeomorphic, i.e., the position of $a_1, \dots, a_n$ in $\mathbb{R}^2$ does not affect the topological type of $\mathbb{R}^2 \smallsetminus \{a_1, \dots, a_n\}$.

11.24. Let $D_1, \dots, D_n \subset \mathbb{R}^2$ be n pairwise disjoint closed disks. Prove that the complement of their union is homeomorphic to a plane with n punctures.

11.25. Let $D_1, \dots, D_n \subset \mathbb{R}^2$ be pairwise disjoint closed disks. The complement of the union of their interiors is called a ***plane with*** n ***holes***. Prove that any two planes with n holes are homeomorphic, i.e., the location of disks $D_1, \dots, D_n$ does not affect the topological type of $\mathbb{R}^2 \smallsetminus \bigcup_{i=1}^n \operatorname{Int} D_i$.

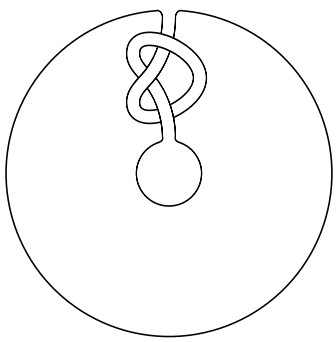
11.26. Let $f, g : \mathbb{R} \to \mathbb{R}$ be two continuous functions such that $f < g$. Prove that the "strip" $\{(x,y) \in \mathbb{R}^2 \mid f(x) \leq y \leq g(x)\}$ bounded by their graphs is homeomorphic to the closed strip $\{(x,y) \mid y \in [0,1]\}$.

11.27. Prove that a mug (with a handle) is homeomorphic to a doughnut.

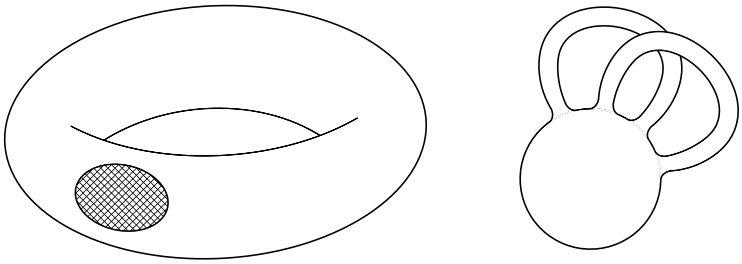
11.28. Arrange the following items to homeomorphism classes: a cup, a saucer, a glass, a spoon, a fork, a knife, a plate, a coin, a nail, a screw, a bolt, a nut, a wedding ring, a drill, a flower pot (with a hole in the bottom), a key.

11.29. In a spherical shell (the space between two concentric spheres), one drilled out a cylindrical hole connecting the boundary spheres. Prove that the rest is homeomorphic to D^3.

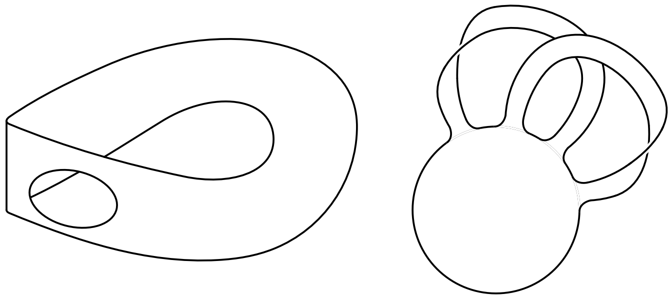
11.30. In a spherical shell, one made a hole connecting the boundary spheres and having the shape of a knotted tube (see Figure below). Prove that the rest of the shell is homeomorphic to D^3.



11.31. Prove that the two surfaces shown in the uppermost Figure on the next page are homeomorphic (they are called ***handles***).



11.32. Prove that the two surfaces shown in the Figure below are homeomorphic. (They are homeomorphic to a *projective plane with two holes*. More details about this is given in Section 22.)



11.33*. Prove that $\mathbb{R}^3 \smallsetminus S^1 \cong \mathbb{R}^3 \smallsetminus \big(\mathbb{R}^1 \cup (0,0,1)\big)$. (What can you say in the case of $\mathbb{R}^n$?)

11.34. Prove that the subset of S^n defined in the standard coordinates in $\mathbb{R}^{n+1}$ by the inequality $x_1^2+x_2^2+\dots+x_k^2 < x_{k+1}^2+\dots+x_n^2$ is homeomorphic to $\mathbb{R}^n \smallsetminus \mathbb{R}^{n-k}$.

⌈11′6⌋ Examples of Nonhomeomorphic Spaces

11.U. Spaces containing different numbers of points are not homeomorphic.

11.V. A discrete space and a (non-one-point) indiscrete space are not homeomorphic.

11.35. Prove that the spaces $\mathbb{Z}$, $\mathbb{Q}$ (with topology induced from $\mathbb{R}$), $\mathbb{R}$, $\mathbb{R}_{T_1}$, and the arrow are pairwise non-homeomorphic.

11.36. Find two spaces X and Y that are not homeomorphic, but there exist continuous bijections $X \to Y$ and $Y \to X$.

⌈11′7⌋ Homeomorphism Problem and Topological Properties

One of the classical problems in topology is the *homeomorphism problem:* to find out whether or not two given topological spaces are homeomorphic. In each special case, the character of solution depends mainly on the answer. In order to prove that two spaces are homeomorphic, it suffices to present a homeomorphism between them. This is essentially what one usually does in this case (and what we did considering all examples of homeomorphic spaces above). However, to prove that two spaces are **not** homeomorphic, it does not suffice to consider any special map, and usually it is impossible to review all the maps. Therefore, proving the nonexistence of a homeomorphism must involve indirect arguments. In particular, we may look for a property or a characteristic shared by homeomorphic spaces and such that one of the spaces has it, while the other one does not. Properties and characteristics that are shared by homeomorphic spaces are called *topological properties* and *invariants*. Obvious examples here are the cardinality (i.e., the number of elements) of the set of points and the set of open sets (cf. Problems *11.34* and *11.U*). Less obvious properties are the main object of the next chapter.

⌈11′8⌋ Information: Nonhomeomorphic Spaces

Euclidean spaces of different dimensions are not homeomorphic. The disks D^p and D^q with $p \neq q$ are not homeomorphic. The spheres S^p and S^q with $p \neq q$ are not homeomorphic. Euclidean spaces are homeomorphic neither to balls, nor to spheres (of any dimension). Letters A and P are not homeomorphic (if the lines are absolutely thin!). The punctured plane $\mathbb{R}^2 \smallsetminus (0,0)$ is not homeomorphic to the plane with a hole, $\mathbb{R}^2 \smallsetminus \{ x^2+y^2 < 1 \}$.

These statements are of different degrees of difficulty. Some of them are considered in the next section. However, some of them cannot be proved by techniques of this course. (See, e.g., [**2**].)

⌈11′9⌋ Embeddings

A continuous map $f : X \to Y$ is a (*topological*) *embedding* if the submap $\text{ab}(f) : X \to f(X)$ is a homeomorphism.

11.W. The inclusion of a subspace into a space is an embedding.

11.X. Composition of embeddings is an embedding.

11.Y. Give an example of a continuous injection which is not a topological embedding. (Find such an example above and create a new one.)

11.37. Find two topological spaces X and Y such that X can be embedded in Y, Y can be embedded in X, but $X \not\cong Y$.

11.38. Prove that $\mathbb{Q}$ cannot be embedded in $\mathbb{Z}$.

11.39. 1) Can a discrete space be embedded in an indiscrete space? 2) What about vice versa?

11.40. Prove that the spaces $\mathbb{R}$, $\mathbb{R}_{T_1}$, and the arrow cannot be embedded in each other.

11.41 Corollary of Inverse Function Theorem. Deduce the following statement from the Inverse Function Theorem (see, e.g., any course of advanced calculus):

Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a continuously differentiable map whose Jacobian $\det(\partial f_i/\partial x_j)$ does not vanish at the origin $0 \in \mathbb{R}^n$. Then the origin has a neighborhood U such that the restriction $f|_U : U \to \mathbb{R}^n$ is an embedding and $f(U)$ is open.

It is of interest that if $U \subset \mathbb{R}^n$ is an open set, then any continuous injection $f : U \to \mathbb{R}^n$ is an embedding and $f(U)$ is also open in $\mathbb{R}^n$. (Certainly, this also implies that $\mathbb{R}^m$ and $\mathbb{R}^n$ with $m \neq n$ are not homeomorphic.)

⌈11′10⌋ Equivalence of Embeddings

Two embeddings $f_1, f_2 : X \to Y$ are ***equivalent*** if there exist homeomorphisms $h_X : X \to X$ and $h_Y : Y \to Y$ such that $f_2 \circ h_X = h_Y \circ f_1$. (The latter equality may be stated as follows: the diagram

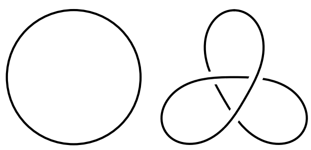
$$\begin{tikzcd} X \arrow[r, "f_1"] \arrow[d, "h_X"'] & Y \arrow[d, "h_Y"] \\ X \arrow[r, "f_2"] & Y \end{tikzcd}$$

is commutative.)

An embedding $S^1 \to \mathbb{R}^3$ is called a ***knot***.

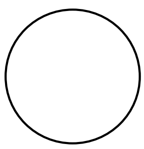
11.42. Prove that any two knots $f_1, f_2 : S^1 \to \mathbb{R}^3$ with $f_1(S^1) = f_2(S^1)$ are equivalent.

11.43. Prove that two knots with images

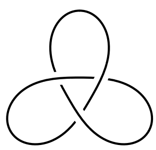


are equivalent.

Information: There are nonequivalent knots. For instance, those with images



and



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Chapter III

Topological Properties

12. Connectedness

⌈12′1⌋ Definitions of Connectedness and First Examples

A topological space X is *connected* if X has only two subsets that are both open and closed: the empty set $\varnothing$ and the entire X. Otherwise, X is *disconnected*.

A *partition* of a set is a cover of this set with pairwise disjoint subsets. To *partition* a set means to construct such a cover.

12.A. *A topological space is connected,*
iff it does not admit a partition into two nonempty open sets,
iff it does not admit a partition into two nonempty closed sets.

12.1. 1) Is an indiscrete space connected? The same question for 2) the arrow and 3) $\mathbb{R}_{T_1}$.

12.2. Describe explicitly all connected discrete spaces.

12.3. Describe explicitly all disconnected two-element spaces.

12.4. 1) Is the set $\mathbb{Q}$ of rational numbers (with the relative topology induced from $\mathbb{R}$) connected? 2) The same question for the set $\mathbb{R} \smallsetminus \mathbb{Q}$ of irrational numbers.

12.5. Let Ω_1 and Ω_2 be two topologies in a set X, and let Ω_2 be finer than Ω_1 (i.e., $\Omega_1 \subset \Omega_2$). 1) If (X, Ω_1) is connected, is (X, Ω_2) connected? 2) If (X, Ω_2) is connected, is (X, Ω_1) connected?

⌈12′2⌋ Connected Sets

When we say that a set A is connected, we mean that A lies in some topological space (which should be clear from the context) and, equipped with the relative topology, A is a connected space.

12.6. Characterize disconnected subsets without mentioning the relative topology.

12.7. Is the set $\{0, 1\}$ connected 1) in $\mathbb{R}$, 2) in the arrow, 3) in $\mathbb{R}_{T_1}$?

12.8. Describe explicitly all connected subsets 1) of the arrow, 2) of $\mathbb{R}_{T_1}$.

12.9. Show that the set $[0, 1] \cup (2, 3]$ is disconnected in $\mathbb{R}$.

12.10. Prove that every nonconvex subset of the real line is disconnected. (In other words, each connected subset of the real line is a singleton or an interval.)

12.11. Let A be a subset of a space X. Prove that A is disconnected iff A has two nonempty subsets B and C such that $A = B \cup C$, $B \cap \operatorname{Cl}_X C = \varnothing$, and $C \cap \operatorname{Cl}_X B = \varnothing$.

12.12. Find a space X and a disconnected subset $A \subset X$ such that if U and V are any two open sets partitioning X, then we have either $U \supset A$, or $V \supset A$.

12.13. Prove that for every disconnected set A in $\mathbb{R}^n$ there are disjoint open sets $U, V \subset \mathbb{R}^n$ such that $A \subset U \cup V$, $U \cap A \neq \varnothing$, and $V \cap A \neq \varnothing$.

Compare *12.11–12.13* with *12.6*.

⌈12′3⌋ Properties of Connected Sets

12.14. Let X be a space. If a set $M \subset X$ is connected and $A \subset X$ is open-closed, then either $M \subset A$, or $M \subset X \smallsetminus A$.

12.B. *The closure of a connected set is connected.*

12.15. Prove that if a set A is connected and $A \subset B \subset \operatorname{Cl} A$, then B is connected.

12.C. *Let $\{A_\lambda\}_{\lambda \in \Lambda}$ be a family of connected subsets of a space X. Assume that any two sets in this family have nonempty intersection. Then $\bigcup_{\lambda \in \Lambda} A_\lambda$ is connected. (In other words: the union of pairwise intersecting connected sets is connected.)*

12.D Special case. If $A, B \subset X$ are two connected sets with $A \cap B \neq \varnothing$, then $A \cup B$ is also connected.

12.E. Let $\{A_\lambda\}_{\lambda \in \Lambda}$ be a family of connected subsets of a space X. Assume that each set in this family meets A_{λ_0} for some $\lambda_0 \in \Lambda$. Then $\bigcup_{\lambda \in \Lambda} A_\lambda$ is connected.

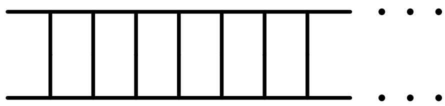
12.F. Let $\{A_k\}_{k \in \mathbb{Z}}$ be a family of connected sets such that $A_k \cap A_{k+1} \neq \varnothing$ for each $k \in \mathbb{Z}$. Prove that $\bigcup_{k \in \mathbb{Z}} A_k$ is connected.

12.16. Let A and B be two connected sets such that $A \cap \operatorname{Cl} B \neq \varnothing$. Prove that $A \cup B$ is also connected.

12.17. Let A be a connected subset of a connected space X, and let $B \subset X \smallsetminus A$ be an open-closed set in the relative topology of $X \smallsetminus A$. Prove that $A \cup B$ is connected.

12.18. Does the connectedness of $A \cup B$ and $A \cap B$ imply that of A and B?

12.19. Let A and B be two sets such that both their union and intersection are connected. Prove that A and B are connected if both of them are 1) open or 2) closed.



12.20. Let $A_1 \supset A_2 \supset \dots$ be an infinite decreasing sequence of closed connected sets in the plane $\mathbb{R}^2$. Is $\bigcap_{k=1}^{\infty} A_k$ a connected set?

⌈12′4⌋ Connected Components

A *connected component* of a space X is a maximal connected subset of X, i.e., a connected subset that is not contained in any other (strictly) larger connected subset of X.

12.G. *Every point belongs to some connected component. Furthermore, this component is unique. It is the union of all connected sets containing this point.*

12.H. *Two connected components either are disjoint or coincide.*

A connected component of a space X is also called just a *component* of X. Theorems *12.G* and *12.H* mean that connected components constitute a partition of the whole space. The next theorem describes the corresponding equivalence relation.

12.I. *Prove that two points lie in the same component iff they belong to the same connected set.*

12.J Corollary. *A space is connected iff any two of its points belong to the same connected set.*

12.K. *Connected components are closed.*

12.21. If each point of a space X has a connected neighborhood, then each connected component of X is open.

12.22. Let x and y belong to the same component. Prove that any open-closed set contains either both x and y, or none of them (cf. *12.37*).

⌈12′5⌋ Totally Disconnected Spaces

A topological space is *totally disconnected* if all of its components are singletons.

12.L Obvious Example. Any discrete space is totally disconnected.

12.M. The space $\mathbb{Q}$ (with the topology induced from $\mathbb{R}$) is totally disconnected.

Note that $\mathbb{Q}$ is not discrete.

12.23. Give an example of an uncountable closed totally disconnected subset of the line.

12.24. Prove that Cantor set (see *2.Jx*) is totally disconnected.

⌈12′6⌋ Boundary and Connectedness

12.25. Prove that if A is a proper nonempty subset of a connected space, then $\operatorname{Fr} A \neq \varnothing$.

12.26. Let F be a connected subset of a space X. Prove that if $A \subset X$ and neither $F \cap A$, nor $F \cap (X \smallsetminus A)$ is empty, then $F \cap \operatorname{Fr} A \neq \varnothing$.

12.27. Let A be a subset of a connected space. Prove that if $\operatorname{Fr} A$ is connected, then so is $\operatorname{Cl} A$.

12.28. Let X be a connected topological space, $U, V \subset X$ two non-disjoint open subsets none of which contains the other one. Prove that if their boundaries $\operatorname{Fr} U$ and $\operatorname{Fr} V$ are connected, then $\operatorname{Fr} U \cap \operatorname{Fr} V \neq \emptyset$

⌈12′7⌋ Connectedness and Continuous Maps

A *continuous image* of a space is its image under a continuous map.

12.N. *A continuous image of a connected space is connected. (In other words, if $f : X \to Y$ is a continuous map and X is connected, then $f(X)$ is also connected.)*

12.O Corollary. *Connectedness is a topological property.*

12.P Corollary. *The number of connected components is a topological invariant.*

12.Q. *A space X is disconnected iff there is a continuous surjection $X \to S^0$.*

12.29. Theorem *12.Q* often yields short proofs of various results concerning connected sets. Apply it for proving, e.g., Theorems *12.B–12.F* and Problems *12.D* and *12.16*.

12.30. Let X be a connected space, $f : X \to \mathbb{R}$ a continuous function. Then $f(X)$ is an interval of $\mathbb{R}$.

12.31. Suppose a space X has a group structure and the multiplication by any element of the group (both from the left and from the right) is a continuous map $X \to X$. Prove that the component of unity is a normal subgroup.

⌈12′8⌋ Connectedness on Line

12.R. *The segment $I = [0,1]$ is connected.*

There are several ways to prove Theorem *12.R*. One of them is suggested by *12.Q*, but refers to the famous Intermediate Value Theorem from Calculus, see *13.A*. However, when studying topology, it would be more natural to find an independent proof and deduce the Intermediate Value Theorem from Theorems *12.R* and *12.Q*. Two problems below provide a sketch of basically the same proof of *12.R*. Cf. *2.Ix* above.

12.R.1 Bisection Method. Let U and V be two subsets of I such that $V = I \smallsetminus U$. Let $a \in U$, $b \in V$, and $a < b$. Prove that there exists a nondecreasing sequence a_n with $a_1 = a$, $a_n \in U$, and a nonincreasing sequence b_n with $b_1 = b$, $b_n \in V$, such that $b_n - a_n = (b-a)/2^{n-1}$.

12.R.2. Under assumptions of *12.R.1*, if U and V are closed in I, then which of them contains $c = \sup\{a_n\} = \inf\{b_n\}$?

12.32. Deduce *12.R* from the result of Problem *2.Ix*.

12.S. Prove that every open set in $\mathbb{R}$ has countably many connected components.

12.T. Prove that $\mathbb{R}^1$ is connected.

12.U. Each convex set in $\mathbb{R}^n$ is connected. (In particular, so are $\mathbb{R}^n$ itself, the ball B^n, and the disk D^n.)

12.V Corollary. Intervals in $\mathbb{R}^1$ are connected.

12.W. Every star-shaped set in $\mathbb{R}^n$ is connected.

12.X Connectedness on Line. *A subset of a line is connected iff it is an interval.*

12.Y. Describe explicitly all nonempty connected subsets of the real line.

12.Z. Prove that the n-sphere S^n is connected. In particular, the circle S^1 is connected.

12.33. Consider the union of the spiral

$$r = \exp\left(\frac{1}{1+\varphi^2}\right), \text{ with } \varphi \geq 0$$

(r, φ are the polar coordinates) and the circle S^1. 1) Is this set connected? 2) Will the answer change if we replace the entire circle by one of its subsets? (Cf. *12.15*.)

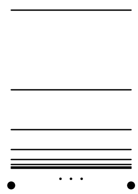
12.34. Are the following subsets of the plane $\mathbb{R}^2$ connected:

(1) the set of points with both coordinates rational;
(2) the set of points with at least one rational coordinate;
(3) the set of points whose coordinates are either both irrational, or both rational?

12.35. Prove that for any $\varepsilon > 0$ the ε-neighborhood of a connected subset of the Euclidean space is connected.

12.36. Prove that each neighborhood U of a connected subset A of the Euclidean space contains a connected neighborhood of A.

12.37. Find a space X and two points belonging to distinct components of X such that each subset $A \subset X$ that is simultaneously open and closed contains either both points, or neither of them. (Cf. *12.22.*)



13. Application of Connectedness

⌈13′1⌋ Intermediate Value Theorem and Its Generalizations

The following theorem is usually included in Calculus. You can easily deduce it from the material of this section. In fact, in a sense it is equivalent to connectedness of the segment.

13.A Intermediate Value Theorem. *A continuous function*

$$f : [a, b] \to \mathbb{R}$$

takes every value between $f(a)$ and $f(b)$.

Many problems that can be solved by using the Intermediate Value Theorem can be found in Calculus textbooks. Here are few of them.

13.1. Prove that any polynomial of odd degree in one variable with real coefficients has at least one real root.

13.B Generalization of 13.A. Let X be a connected space, $f : X \to \mathbb{R}$ a continuous function. Then $f(X)$ is an interval of $\mathbb{R}$.

13.C Corollary. Let $J \subset \mathbb{R}$ be an interval of the real line, $f : J \to \mathbb{R}$ a continuous function. Then $f(J)$ is also an interval of $\mathbb{R}$. (In other words, continuous functions map intervals to intervals.)

⌈13′2⌋ Applications to Homeomorphism Problem

Connectedness is a topological property, and the number of connected components is a topological invariant (see Section 11).

13.D. $[0, 2]$ and $[0, 1] \cup [2, 3]$ are not homeomorphic.

Simple constructions assigning homeomorphic spaces to homeomorphic ones (e.g., deleting one or several points), allow us to use connectedness for proving that some *connected* spaces are not homeomorphic.

13.E. I, $[0, \infty)$, $\mathbb{R}^1$, and S^1 are pairwise nonhomeomorphic.

13.2. Prove that a circle is not homeomorphic to a subspace of $\mathbb{R}^1$.

13.3. Give a topological classification of the letters of the alphabet: A, B, C, D, . . . , regarded as subsets of the plane (the arcs comprising the letters are assumed to have zero thickness).

13.4. Prove that square and segment are not homeomorphic.

Recall that there exist continuous surjections of the segment onto square, which are called *Peano curves*, see Section 10.

13.F. $\mathbb{R}^1$ and $\mathbb{R}^n$ are not homeomorphic if $n > 1$.

Information. $\mathbb{R}^p$ and $\mathbb{R}^q$ are not homeomorphic unless $p = q$. This follows, for instance, from the Lebesgue–Brouwer Theorem on the invariance of dimension (see, e.g., W. Hurewicz and H. Wallman, *Dimension Theory*, Princeton, NJ, 1941).

13.5. The statement "$\mathbb{R}^p$ is not homeomorphic to $\mathbb{R}^q$ unless $p = q$" implies that S^p is not homeomorphic to S^q unless $p = q$.

⌈13′3x⌋ Induction on Connectedness

A map $f : X \to Y$ is ***locally constant*** if each point of X has a neighborhood U such that the restriction of f to U is constant.

13.6x. Prove that any locally constant map is continuous.

13.7x. A locally constant map on a connected set is constant.

13.8x. Riddle. How are *12.26* and *13.7x* related?

13.9x. Let G be a group equipped with a topology such that for each $g \in G$ the map $G \to G : x \mapsto xgx^{-1}$ is continuous, and let G with this topology be connected. Prove that if the topology induced on a normal subgroup H of G is discrete, then H is contained in the center of G (i.e., $hg = gh$ for any $h \in H$ and $g \in G$).

13.10x Induction on Connectedness. Let $\mathcal{E}$ be a property of subsets of a topological space X such that the union of sets with nonempty pairwise intersections inherits this property from the sets involved. Prove that if X is connected and each point in X has a neighborhood with property $\mathcal{E}$, then X also has property $\mathcal{E}$.

13.11x. Prove *13.7x* and solve *13.9x* using *13.10x*.

For more applications of induction on connectedness, see *14.T*, *14.22x*, *14.24x*, and *14.26x*.

⌈13′4x⌋ Dividing Pancakes

13.12x. Any irregularly shaped pancake can be cut in half by one stroke of the knife made in any prescribed direction. In other words, if A is a bounded open set in the plane and l is a line in the plane, then a certain line L parallel to l divides A in half by area.

13.13x. If, under the assumptions of *13.12x*, A is connected, then L is unique.

13.14x. Suppose two irregularly shaped pancakes lie on the same platter; show that it is possible to cut both exactly in half by one stroke of the knife. In other words: if A and B are two bounded regions in the plane, then there exists a line in the plane that bisects the area of each of the regions.

13.15x. Prove that a plane pancake of any shape can be divided into four pieces of equal area by two mutually perpendicular straight cuts. In other words, if A is a bounded connected open set in the plane, then there are two perpendicular lines that divide A into four parts having equal areas.

13.16x. Riddle. What if the knife is curved and makes cuts of a shape different from the straight line? For what shapes of the cuts can you formulate and solve problems similar to *13.12x*–*13.15x*?

13.17x. Riddle. Formulate and solve counterparts of Problems *13.12x*–*13.15x* for regions in three-space. Can you increase the number of regions in the counterparts of *13.12x* and *13.14x*?

13.18x. Riddle. What about pancakes in $\mathbb{R}^n$?

14. Path Connectedness

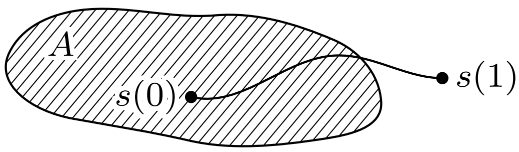
⌈14′1⌋ Paths

A *path* in a topological space X is a continuous map of the segment $I = [0,1]$ to X. The point $s(0)$ is the *initial* point of a path $s : I \to X$, while $s(1)$ is the *final* point of s. We say that the path s *connects* $s(0)$ with $s(1)$. This terminology is inspired by an image of a moving point: at the moment $t \in [0,1]$, the point is at $s(t)$.

To tell the truth, this is more than what is usually called a path, since, besides information on the trajectory of the point, it contains a complete account of the movement: the schedule saying when the point goes through each point.

14.1. If $s : I \to X$ is a path, then the image $s(I) \subset X$ is connected.

14.2. Let $s : I \to X$ be a path connecting a point in a set $A \subset X$ with a point in $X \setminus A$. Prove that $s(I) \cap \operatorname{Fr}(A) \neq \varnothing$.

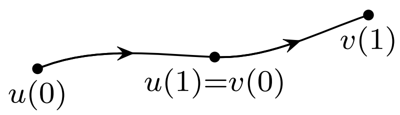


14.3. Let A be a subset of a space X, and let $\mathrm{in}_A : A \to X$ be the inclusion. Prove that $u : I \to A$ is a path in A iff the composition $\mathrm{in}_A \circ u : I \to X$ is a path in X.

A constant map $s_a : I \to X : x \mapsto a$ is a *stationary* path. Each path s has an *inverse* path $s^{-1} : t \mapsto s(1-t)$. Although, strictly speaking, this notation is already used (for the inverse map), the ambiguity of notation usually leads to no confusion: as a rule, inverse maps do not appear in contexts involving paths.

Let $u : I \to X$ and $v : I \to X$ be two paths such that $u(1) = v(0)$. We define

$$uv : I \to X : t \mapsto \begin{cases} u(2t) & \text{if } t \in [0,1/2], \\ v(2t-1) & \text{if } t \in [1/2,1]. \end{cases} \tag{22}$$



14.A. Prove that the above map $uv : I \to X$ is continuous (i.e., it is a path). Cf. *10.T* and *10.V*.

The path uv is the *product* of u and v. Recall that uv is defined only if the final point $u(1)$ of u is the initial point $v(0)$ of v.

⌈14′2⌋ Path-Connected Spaces

A topological space X is *path-connected* (or *arcwise connected*) if any two points are connected in X by a path.

14.B. Prove that the segment I is path-connected.

14.C. Prove that the Euclidean space of any dimension is path-connected.

14.D. Prove that the n-sphere S^n with $n > 0$ is path-connected.

14.E. Prove that the 0-sphere S^0 is not path-connected.

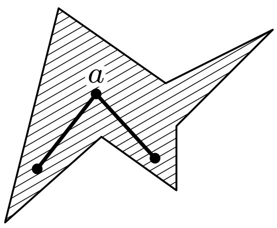
14.4. Which of the following spaces are path-connected:
(1) a discrete space; (2) an indiscrete space;
(3) the arrow; (4) $\mathbb{R}_{T_1}$;
(5) ?

⌈14′3⌋ Path-Connected Sets

A *path-connected set* (or *arcwise connected set*) is a subset of a topological space (which should be clear from the context) that is path-connected as a subspace (the space with the relative topology).

14.5. Prove that a subset A of a space X is path-connected iff any two points in A are connected by a path $s : I \to X$ with $s(I) \subset A$.

14.6. Prove that each convex subset of Euclidean space is path-connected.



14.7. Every star-shaped set in $\mathbb{R}^n$ is path-connected.

14.8. The image of a path is a path-connected set.

14.9. Prove that the set of plane convex polygons with topology generated by the Hausdorff metric is path-connected. (What can you say about the set of convex n-gons with fixed n?)

14.10. ***Riddle.*** What can you say about the assertion of Problem *14.9* in the case of arbitrary (not necessarily convex) polygons?

⌈14′4⌋ Properties of Path-Connected Sets

Path connectedness is very similar to connectedness. Further, in some important situations it is even equivalent to connectedness. However, some properties of connectedness do not carry over to the case of path connectedness (see *14.Q* and *14.R*). For the properties that do carry over, proofs are usually easier in the case of path connectedness.

14.F. *The union of a family of pairwise intersecting path-connected sets is path-connected.*

14.11. Prove that if two sets A and B are both closed or both open and their union and intersection are path-connected, then A and B are also path-connected.

14.12. 1) Prove that the interior and boundary of a path-connected set may be not path-connected. 2) Connectedness shares this property.

14.13. Let A be a subset of the Euclidean space. Prove that if $\operatorname{Fr} A$ is path-connected, then so is $\operatorname{Cl} A$.

14.14. Prove that the same holds true for a subset of an arbitrary path-connected space.

⌈14′5⌋ Path-Connected Components

A *path-connected component* or *arcwise connected component* of a space X is a path-connected subset of X that is not contained in any other path-connected subset of X.

14.G. *Every point belongs to a path-connected component.*

14.H. *Two path-connected components either coincide or are disjoint.*

Theorems *14.G* and *14.H* mean that path-connected components constitute a partition of the entire space. The next theorem describes the corresponding equivalence relation.

14.I. Prove that two points belong to the same path-connected component iff they are connected by a path (cf. *12.I*).

Unlike the case of connectedness, path-connected components are not necessarily closed. (See *14.Q*, cf. *14.P* and *14.R*.)

⌈14′6⌋ Path Connectedness and Continuous Maps

14.J. *A continuous image of a path-connected space is path-connected.*

14.K Corollary. *Path connectedness is a topological property.*

14.L Corollary. *The number of path-connected components is a topological invariant.*

⌈14′7⌋ Path Connectedness Versus Connectedness

14.M. *Any path-connected space is connected.*

Put

$$A = \{ (x, y) \in \mathbb{R}^2 \mid x > 0,\ y = \sin(1/x) \}, \quad X = A \cup (0, 0).$$

14.15. Sketch A.

14.N. Prove that A is path-connected and X is connected.

14.O. Prove that deleting any point from A makes A and X disconnected (and, hence, not path-connected).

14.P. *X is not path-connected.*

14.Q. Find an example of a path-connected set whose closure is not path-connected.

14.R. Find an example of a path-connected component that is not closed.

14.S. *If each point of a space X has a path-connected neighborhood, then each path-connected component of X is open. (Cf. 12.21.)*

14.T. *Assume that each point of a space X has a path-connected neighborhood. Then X is path-connected iff X is connected.*

14.U. *For open subsets of the Euclidean space, connectedness is equivalent to path connectedness.*

14.16. For subsets of the real line, path connectedness and connectedness are equivalent.

14.17. Prove that for each $\varepsilon > 0$ the ε-neighborhood of a connected subset of the Euclidean space is path-connected.

14.18. Prove that each neighborhood U of a connected subset A of the Euclidean space contains a path-connected neighborhood of A.

⌈14′8x⌋ Polyline-Connectedness

A subset A of Euclidean space is *polyline-connected* if any two points of A are joined by a finite broken line (a *polyline*) contained in A.

14.19x. Each polyline-connected set in $\mathbb{R}^n$ is path-connected, and thus also connected.

14.20x. Each convex set in $\mathbb{R}^n$ is polyline-connected.

14.21x. Each star-shaped set in $\mathbb{R}^n$ is polyline-connected.

14.22x. Prove that for open subsets of the Euclidean space connectedness is equivalent to polyline-connectedness.

14.23x. Construct a non-one-point path-connected subset A of Euclidean space such that no two distinct points of A are connected by a polyline in A.

***14.24*x.** Let $X \subset \mathbb{R}^2$ be a countable set. Prove that $\mathbb{R}^2 \setminus X$ is polyline-connected.

***14.25*x.** Let $X \subset \mathbb{R}^n$ be the union of countably many affine subspaces with dimensions at most $n-2$. Prove that $\mathbb{R}^n \setminus X$ is polyline-connected.

***14.26*x.** Let $X \subset \mathbb{C}^n$ be the union of countably many algebraic subsets (i.e., subsets defined by systems of algebraic equations in the standard coordinates of $\mathbb{C}^n$). Prove that $\mathbb{C}^n \setminus X$ is polyline-connected.

⌈14′9x⌋ Connectedness of Some Sets of Matrices

Recall that real $n \times n$ matrices constitute a space, which differs from $\mathbb{R}^{n^2}$ only in the way of enumerating its natural coordinates (they are numbered by pairs of indices). The same holds true for the set of complex $n \times n$ matrices and $\mathbb{C}^{n^2}$ (which is homeomorphic to $\mathbb{R}^{2n^2}$).

***14.27*x.** Find connected and path-connected components of the following subspaces of the space of real $n \times n$ matrices:

(1) $GL(n;\mathbb{R}) = \{A \mid \det A \neq 0\}$;
(2) $O(n;\mathbb{R}) = \{A \mid A \cdot ({}^tA) = \mathbb{E}\}$;
(3) $\mathrm{Symm}(n;\mathbb{R}) = \{A \mid {}^tA = A\}$;
(4) $\mathrm{Symm}(n;\mathbb{R}) \cap GL(n;\mathbb{R})$;
(5) $\{A \mid A^2 = \mathbb{E}\}$.

***14.28*x.** Find connected and path-connected components of the following subspaces of the space of complex $n \times n$ matrices:

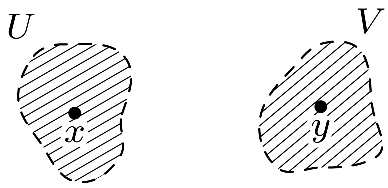
(1) $GL(n;\mathbb{C}) = \{A \mid \det A \neq 0\}$;
(2) $U(n;\mathbb{C}) = \{A \mid A \cdot ({}^t\bar{A}) = \mathbb{E}\}$;
(3) $Herm(n;\mathbb{C}) = \{A \mid {}^tA = \bar{A}\}$;
(4) $Herm(n;\mathbb{C}) \cap GL(n;\mathbb{C})$.

15. Separation Axioms

Our purpose in this section is to consider natural restrictions on the topological structure making the structure closer to being metrizable. They are called "Separation Axioms". A lot of separation axioms are known. We restrict ourselves to the five most important of them. They are numerated, and denoted by T_0, T_1, T_2, T_3, and T_4, respectively.[1]

$\lceil 15'1 \rfloor$ Hausdorff Axiom

We start with the second axiom, which is the most important one. In addition to the designation T_2, it has a name: the *Hausdorff axiom*. A topological space satisfying T_2 is a *Hausdorff space*. This axiom is stated as follows: *any two distinct points possess disjoint neighborhoods*. We can state it more formally: $\forall\, x, y \in X,\ x \neq y\ \exists U_x, V_y :\ U_x \cap V_y = \varnothing$.



15.A. *Any metric space is Hausdorff.*

15.1. Which of the following spaces are Hausdorff:

(1) a discrete space;
(2) an indiscrete space;
(3) the arrow;
(4) $\mathbb{R}_{T_1}$;
(5) ?

If the next problem holds you up even for a minute, we advise you to think over all definitions and solve all simple problems.

15.B. Is the segment $[0, 1]$ with the topology induced from $\mathbb{R}$ a Hausdorff space? Do the points 0 and 1 possess disjoint neighborhoods? Which, if any?

15.C. A space X is Hausdorff iff for each $x \in X$ we have $\{x\} = \bigcap_{U \ni x} \operatorname{Cl} U$.

[1] The letter T in these designations originates from the German word *Trennungsaxiom*, which means *separation axiom*.

⌈15′2⌋ Limits of Sequences

Let $\{a_n\}$ be a sequence of points of a topological space X. A point $b \in X$ is the ***limit*** of the sequence if for any neighborhood U of b there exists a number N such that $a_n \in U$ for any $n \geq N$.[2] In this case, we say that the sequence ***converges*** or ***tends*** to b as n tends to infinity.

15.2. Explain the meaning of the statement "b is not a limit of sequence a_n" by using as few negations (i.e., the words *no*, *not*, *none*, etc.) as you can.

15.3. The limit of a sequence does not depend on the order of the terms. More precisely, let a_n be a convergent sequence: $a_n \to b$, and let $\phi : \mathbb{N} \to \mathbb{N}$ be a bijection. Then the sequence $a_{\phi(n)}$ is also convergent and has the same limit: $a_{\phi(n)} \to b$. For example, if the terms in the sequence are pairwise distinct, then the convergence and the limit depend only on the set of terms, which shows that these notions actually belong to geometry.

15.D. *Any sequence in a Hausdorff space has at most one limit.*

15.E. Prove that each point in the space $\mathbb{R}_{T_1}$ is a limit of the sequence $a_n = n$.

⌈15′3⌋ Coincidence Set and Fixed Point Set

Let $f, g : X \to Y$ be two maps. Then the set $C(f, g) = \{x \in X \mid f(x) = g(x)\}$ is the ***coincidence set*** of f and g.

15.4. Prove that the coincidence set of two continuous maps from an arbitrary space to a Hausdorff space is closed.

15.5. Construct an example proving that the Hausdorff condition in *15.4* is essential.

A point $x \in X$ is a ***fixed point*** of a map $f : X \to X$ if $f(x) = x$. The set of all fixed points of a map f is the ***fixed point set*** of f.

15.6. Prove that the fixed-point set of a continuous map from a Hausdorff space to itself is closed.

15.7. Construct an example showing that the Hausdorff condition in *15.6* is essential.

15.8. Prove that if $f, g : X \to Y$ are two continuous maps, Y is Hausdorff, A is everywhere dense in X, and $f|_A = g|_A$, then $f = g$.

15.9. Riddle. How are Problems *15.4*, *15.6*, and *15.8* related to each other?

⌈15′4⌋ Hereditary Properties

A topological property is ***hereditary*** if it carries over from a space to its subspaces, which means that if a space X has this property, then each subspace of X also has it.

[2] You can also rephrase this as follows: each (arbitrarily small) neighborhood of b contains all members of the sequence that have sufficiently large indices.

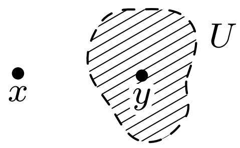
15.10. Which of the following topological properties are hereditary:

(1) finiteness of the set of points;
(2) finiteness of the topological structure;
(3) infiniteness of the set of points;
(4) connectedness;
(5) path connectedness?

15.F. *The property of being a Hausdorff space is hereditary.*

⌈15′5⌋ The First Separation Axiom

A topological space X satisfies the *first separation axiom* T_1 if each one of any two points of X has a neighborhood that does not contain the other point.[3] More formally: $\forall\, x, y \in X,\ x \neq y\ \exists U_y :\ x \notin U_y$.



15.G. *For any topological space X, the following three assertions are equivalent:*

- *the space X satisfies the first separation axiom,*
- *all one-point sets in X are closed,*
- *all finite sets in X are closed.*

15.11. Prove that a space X satisfies the first separation axiom iff every point of X is the intersection of all of its neighborhoods.

15.12. Any Hausdorff space satisfies the first separation axiom.

15.H. Any finite set in a Hausdorff space is closed.

15.I. A metric space satisfies the first separation axiom.

15.13. Find an example showing that the first separation axiom does not imply the Hausdorff axiom.

15.J. Show that $\mathbb{R}_{T_1}$ satisfies the first separation axiom, but is not a Hausdorff space (cf. *15.13*).

15.K. *The first separation axiom is hereditary.*

15.14. Suppose that for any two distinct points a and b of a space X there exists a continuous map f from X to a space with the first separation axiom such that $f(a) \neq f(b)$. Prove that X also satisfies the first separation axiom.

15.15. Prove that a continuous map of an indiscrete space to a space satisfying axiom T_1 is constant.

[3]Axiom T_1 is also called the Tikhonov axiom.

15.16. Prove that every set has the coarsest topological structure satisfying the first separation axiom. Describe this structure.

⌈15′6⌋ The Kolmogorov Axiom

The first separation axiom emerges as a weakened Hausdorff axiom.

15.L. Riddle. How can the first separation axiom be weakened?

A topological space satisfies the *Kolmogorov axiom* or the *zeroth separation axiom* T_0 if *at least one* of any two distinct points of this space has a neighborhood that does not contain the other point.

15.M. An indiscrete space containing at least two points does not satisfy axiom T_0.

15.N. *The following properties of a space X are equivalent:*

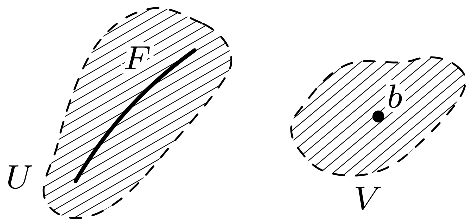
(1) *X satisfies the Kolmogorov axiom;*
(2) *any two different points of X have different closures;*
(3) *X contains no indiscrete subspace consisting of two points.*
(4) *X contains no indiscrete subspace consisting of more than one point.*

15.O. *A topology is a poset topology iff it is a smallest neighborhood topology satisfying the Kolmogorov axiom.*

Thus, on the one hand, posets give rise to numerous examples of topological spaces, among which we see the most important spaces, like the line with the standard topology. On the other hand, all posets are obtained from topological spaces of a special kind, which are quite far away from the class of metric spaces.

⌈15′7⌋ The Third Separation Axiom

A topological space X satisfies the *third separation axiom* if every closed set in X and every point of its complement have disjoint neighborhoods, i.e., for every closed set $F \subset X$ and every point $b \in X \setminus F$ there exist disjoint open sets $U, V \subset X$ such that $F \subset U$ and $b \in V$.



A space is *regular* if it satisfies the first and third separation axioms.

15.P. A regular space is a Hausdorff space.

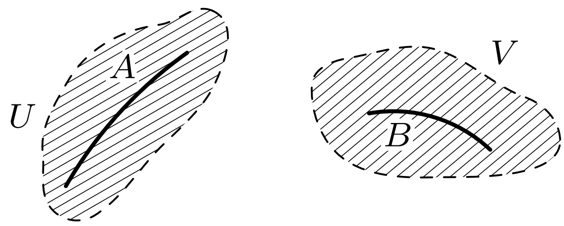
15.Q. A space is regular iff it satisfies the second and third separation axioms.

> ***15.17.*** Find a Hausdorff space which is not regular.
>
> ***15.18.*** Find a space satisfying the third, but not the second separation axiom.
>
> ***15.19.*** Prove that a space X satisfies the third separation axiom iff every neighborhood of every point $x \in X$ contains the closure of a neighborhood of x.
>
> ***15.20.*** Prove that the third separation axiom is hereditary.

15.R. *Any metric space is regular.*

⌈15′8⌋ The Fourth Separation Axiom

A topological space X satisfies the *fourth separation axiom* if any two disjoint closed sets in X have disjoint neighborhoods, i.e., for any two closed sets $A, B \subset X$ with $A \cap B = \varnothing$ there exist open sets $U, V \subset X$ such that $U \cap V = \varnothing$, $A \subset U$, and $B \subset V$.



A space is *normal* if it satisfies the first and fourth separation axioms.

15.S. A normal space is regular (and hence Hausdorff).

15.T. A space is normal iff it satisfies the second and fourth separation axioms.

> ***15.21.*** Find a space which satisfies the fourth, but not second separation axiom.
>
> ***15.22.*** Prove that a space X satisfies the fourth separation axiom iff every neighborhood of every closed set $F \subset X$ contains the closure of some neighborhood of F.
>
> ***15.23.*** Prove that each closed subspace of a normal space is normal.
>
> ***15.24.*** Let X satisfy the fourth separation axiom, and let $F_1, F_2, F_3 \subset X$ be three closed subsets with empty intersection: $F_1 \cap F_2 \cap F_3 = \emptyset$. Prove that they have neighborhoods U_1, U_2, U_3 with empty intersection.

15.U. *Any metric space is normal.*

> ***15.25.*** Find two closed disjoint subsets A and B of some metric space such that $\inf\{\rho(a, b) \mid a \in A,\ b \in B\} = 0$.
>
> ***15.26.*** Let $f : X \to Y$ be a continuous surjection such that the image of each closed set is closed. Prove that if X is normal, then so is Y.

⌈15′9x⌋ Nemytskii's Space

Denote by $\mathcal{H}$ the open upper half-plane $\{(x, y) \in \mathbb{R}^2 \mid y > 0\}$ equipped with the topology generated by the Euclidean metric. Denote by $\mathcal{N}$ the union of $\mathcal{H}$ and the boundary line $\mathbb{R}^1$: $\mathcal{N} = \mathcal{H} \cup \mathbb{R}^1$, but equip it with the topology obtained by adjoining to the Euclidean topology the sets of the form $x \cup D$, where $x \in \mathbb{R}^1$ and D is an open disk in $\mathcal{H}$ touching $\mathbb{R}^1$ at the point x. This is the *Nemytskii space*. It can be used to clarify properties of the fourth separation axiom.

***15.27*x.** Prove that the Nemytskii space is Hausdorff.

***15.28*x.** Prove that the Nemytskii space is regular.

***15.29*x.** What topological structure is induced on $\mathbb{R}^1$ from $\mathcal{N}$?

***15.30*x.** Prove that the Nemytskii space is not normal.

15.31*x *Corollary. There exists a regular space which is not normal.

***15.32*x.** Embed the Nemytskii space in a normal space in such a way that the complement of the image would be a single point.

15.33*x *Corollary. Theorem *15.23* does not extend to nonclosed subspaces, i.e., the property of being normal is not hereditary, is it?

⌈15′10x⌋ Urysohn Lemma and Tietze Theorem

***15.34*x.** Let A and B be two disjoint closed subsets of a metric space X. Then there exists a continuous function $f : X \to I$ such that $f^{-1}(0) = A$ and $f^{-1}(1) = B$.

***15.35*x.** Let F be a closed subset of a metric space X. Then any continuous function $f : X \to [-1, 1]$ extends over the whole X.

15.35*x*.1. Let F be a closed subset of a metric space X. For any continuous function $f : F \to [-1, 1]$, there exists a function $g : X \to \big[-1/3, 1/3\big]$ such that $|f(x) - g(x)| \le 2/3$ for each $x \in F$.

15.V*x *Urysohn Lemma. *Let A and B be two nonempty disjoint closed subsets of a normal space X. Then there exists a continuous function $f : X \to I$ such that $f(A) = 0$ and $f(B) = 1$.*

15.V*x*.1. Let A and B be two disjoint closed subsets of a normal space X. Consider the set $\Lambda = \big\{\frac{k}{2^n} \mid k, n \in \mathbb{Z}_+,\ k \le 2^n\big\}$. There exists a collection $\{U_p\}_{p\in\Lambda}$ of open subsets of X such that for any $p, q \in \Lambda$ we have: 1) $A \subset U_0$ and $B \subset X \smallsetminus U_1$, and 2) if $p < q$, then $\operatorname{Cl} U_p \subset U_q$.

15.W*x *Tietze Extension Theorem. *Let A be a closed subset of a normal space X. Let $f : A \to [-1, 1]$ be a continuous function. Prove that there exists a continuous function $F : X \to [-1, 1]$ such that $F|_A = f$.*

15.X*x *Corollary. Let A be a closed subset of a normal space X. Then any continuous function $A \to \mathbb{R}$ extends to a function on the whole X.

***15.36*x.** Will the statement of the Tietze theorem remain true if we replace the segment $[-1, 1]$ in the hypothesis by $\mathbb{R}$, $\mathbb{R}^n$, S^1, or S^2?

***15.37*x.** Derive the Urysohn Lemma from the Tietze Extension Theorem.

16. Countability Axioms

In this section, we continue to study topological properties that are additionally imposed on a topological structure in order to make the abstract situation under consideration closer to special situations and hence richer in contents. The restrictions studied in this section bound a topological structure "from above": they require that something be countable.

⌈16′1⌋ Set-Theoretic Digression: Countability

Recall that two sets have equal *cardinality* if there exists a bijection of one of them onto the other. A set of the same cardinality as a subset of the set $\mathbb{N}$ of positive integers is *countable*.

16.1. A set X is countable iff there exists an injection $X \to \mathbb{N}$ (or, more generally, an injection of X into another countable set).

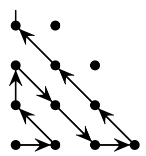
Sometimes this term is used only for infinite countable sets, i.e., for sets of the cardinality of the whole set $\mathbb{N}$ of positive integers, while sets countable in the above sense are said to be *at most countable*. This is less convenient. In particular, if we adopted this terminology, this section would be called "At Most Countability Axioms". This would also lead to other more serious inconveniences as well. Our terminology has the following advantageous properties.

16.A. Any subset of a countable set is countable.

16.B. The image of a countable set under any map is countable.

16.C. The following sets are countable:

(1) $\mathbb{Z}$,
(2) $\mathbb{N}^2 = \{(k, n) \mid k, n \in \mathbb{N}\}$,
(3) $\mathbb{Q}$.



16.D. The union of a countable family of countable sets is countable.

16.E. $\mathbb{R}$ is not countable.

16.2. Prove that each set Σ of disjoint figure-eight curves in the plane is countable.

⌈16′2⌋ Second Countability and Separability

In this section, we study three restrictions on the topological structure. Two of them have numbers (one and two), the third one has no number. As in the previous section, we start from the restriction having number two.

A topological space X satisfies the *second axiom of countability* or is *second countable* if X has a countable base. A space is *separable* if it contains a countable dense set. (This is the countability axiom without a number that we mentioned above.)

16.F. *The second axiom of countability implies separability.*

16.G. The second axiom of countability is hereditary.

> ***16.3.*** Are the arrow and $\mathbb{R}_{T_1}$ second countable?
>
> ***16.4.*** Are the arrow and $\mathbb{R}_{T_1}$ separable?
>
> ***16.5.*** Construct an example proving that separability is not hereditary.

16.H. *A metric separable space is second countable.*

16.I Corollary. *For metrizable spaces, separability is equivalent to the second axiom of countability.*

16.J. (Cf. *16.5.*) Prove that for metrizable spaces separability is hereditary.

16.K. Prove that Euclidean spaces and all their subspaces are separable and second countable.

> ***16.6.*** Construct a metric space which is not second countable.
>
> ***16.7.*** Prove that each collection of pairwise disjoint open sets in a separable space is countable.
>
> ***16.8.*** Prove that the set of components of an open set $A \subset \mathbb{R}^n$ is countable.

16.L. *A continuous image of a separable space is separable.*

> ***16.9.*** Construct an example proving that a continuous image of a second countable space may be not second countable.

16.M Lindelöf Theorem. *Any open cover of a second countable space contains a countable part that also covers the space.*

> ***16.10.*** Prove that each base of a second countable space contains a countable part which is also a base.
>
> ***16.11 Brouwer Theorem*.*** Let $\{K_\lambda\}$ be a family of closed sets of a second countable space and assume that for every decreasing sequence $K_1 \supset K_2 \supset \dots$ of sets in this family the intersection $\bigcap_{n=1}^{\infty} K_n$ also belongs to the family. Then the family contains a minimal set A, i.e., a set such that no proper subset of A belongs to the family.

⌈16′3⌋ Bases at a Point

Let X be a space, a a point of X. A *neighborhood base* at a or just a *base of* X *at* a is a collection Σ of neighborhoods of a such that each neighborhood of a contains a neighborhood from Σ.

16.N. If Σ is a base of a space X, then $\{U \in \Sigma \mid a \in U\}$ is a base of X at a.

16.12. In a metric space, the following collections of balls are neighborhood bases at a point a:

- the set of all open balls with center a;
- the set of all open balls with center a and rational radii;
- the set of all open balls with center a and radii r_n, where $\{r_n\}$ is any sequence of positive numbers converging to zero.

16.13. What are the minimal bases at a point in the discrete and indiscrete spaces?

⌈16′4⌋ First Countability

A topological space X satisfies the *first axiom of countability* or is a *first countable space* if X has a countable neighborhood base at each point.

16.O. *Any metric space is first countable.*

16.P. *The second axiom of countability implies the first one.*

16.Q. Find a first countable space which is not second countable. (Cf. *16.6*.)

16.14. Which of the following spaces are first countable:
(1) the arrow; (2) $\mathbb{R}_{T_1}$;
(3) a discrete space; (4) an indiscrete space?

16.15. Find a first countable separable space which is not second countable.

16.16. Prove that if X is a first countable space, then at each point it has a decreasing countable neighborhood base: $U_1 \supset U_2 \supset \dots$.

⌈16′5⌋ Sequential Approach to Topology

Specialists in Mathematical Analysis love sequences and their limits. Moreover, they like to talk about all topological notions by relying on the notions of sequence and its limit. This tradition has little mathematical justification, except for a long history descending from the XIXth century's studies on the foundations of analysis. In fact, almost always[4] it is more convenient to avoid sequences, provided that you deal with topological notions, except summation of series, where sequences are involved in the underlying definitions. Paying a tribute to this tradition, here we explain how and in

[4]The exceptions which one may find in the standard curriculum of a mathematical department can be counted on two hands.

what situations topological notions can be described in terms of sequences and their limits.

Let A be a subset of a space X. The set $\operatorname{SCl} A$ of limits of all sequences a_n with $a_n \in A$ is the ***sequential closure*** of A.

16.R. Prove that $\operatorname{SCl} A \subset \operatorname{Cl} A$.

16.S. *If a space X is first countable, then the opposite inclusion* $\operatorname{Cl} A \subset \operatorname{SCl} A$ *also holds true for each* $A \subset X$, *whence* $\operatorname{SCl} A = \operatorname{Cl} A$.

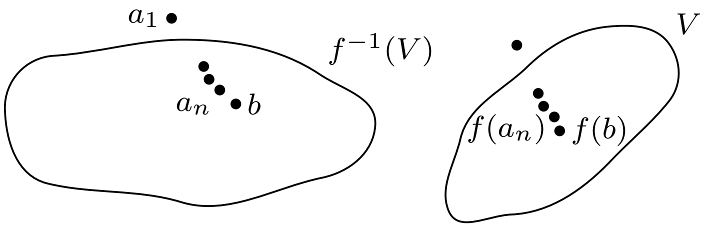
Therefore, in a first countable space (in particular, in any metric space) we can recover (hence, define) the closure of a set provided that we know which sequences are convergent and what their limits are. In turn, the knowledge of closures allows one to determine which sets are closed. As a consequence, knowledge of closed sets allows one to recover open sets and all other topological notions.

> ***16.17.*** Let X be the set of real numbers equipped with the topology consisting of $\varnothing$ and complements of all countable subsets. (Check that this is actually a topology.) Describe convergent sequences, sequential closure and closure in X. Prove that X contains a set A with $\operatorname{SCl} A \neq \operatorname{Cl} A$.

⌈16′6⌋ Sequential Continuity

Now we consider the continuity of maps along the same lines. A map $f : X \to Y$ is ***sequentially continuous*** if for each $b \in X$ and each sequence $a_n \in X$ converging to b the sequence $f(a_n)$ converges to $f(b)$.

16.T. *Any continuous map is sequentially continuous.*



16.U. *The preimage of a sequentially closed set under a sequentially continuous map is sequentially closed.*

16.V. *If X is a first countable space, then any sequentially continuous map $f : X \to Y$ is continuous.*

Thus, continuity and sequential continuity are equivalent for maps of a first countable space.

> ***16.18.*** Construct a discontinuous map which is sequentially continuous. (Cf. Problem *16.17.*)

⌈16′7x⌋ Embedding and Metrization Theorems

***16.W*x.** Prove that the space l_2 is separable and second countable.

***16.X*x.** Prove that a regular second countable space is normal.

***16.Y*x.** Prove that a normal second countable space can be embedded in l_2. (Use the Urysohn Lemma *15.Vx*.)

***16.Z*x.** Prove that a second countable space is metrizable iff it is regular.

17. Compactness

⌈17′1⌋ Definition of Compactness

This section is devoted to a topological property playing a very special role in topology and its applications. It is a sort of topological counterpart for the property of being finite in the context of set theory. (It seems though that this analogy has never been formalized.)

A topological space X is *compact* if each open cover of X contains a finite part that also covers X.

If Γ is a cover of X and $\Sigma \subset \Gamma$ is a cover of X, then Σ is a *subcover* (or *subcovering*) of Γ. Thus, a space X is compact if every open cover of X contains a finite subcovering.

17.A. Any finite space and indiscrete space are compact.

17.B. Which discrete spaces are compact?

> ***17.1.*** Let $\Omega_1 \subset \Omega_2$ be two topological structures in X. 1) Does the compactness of (X, Ω_2) imply that of (X, Ω_1)? 2) And vice versa?

17.C. The line $\mathbb{R}$ is not compact.

17.D. A space X is not compact iff it has an open cover containing no finite subcovering.

> ***17.2.*** Is the arrow compact? Is $\mathbb{R}_{T_1}$ compact?

⌈17′2⌋ Terminology Remarks

Originally the word *compactness* was used for the following weaker property: any countable open cover contains a finite subcovering.

17.E. For a second countable space, the original definition of compactness is equivalent to the modern one.

The modern notion of compactness was introduced by P. S. Alexandrov (1896–1982) and P. S. Urysohn (1898–1924). They suggested for it the term *bicompactness*. This notion turned out to be fortunate; it has displaced the original one and even took its name, i.e., "compactness". The term bicompactness is sometimes used (mainly by topologists of Alexandrov's school).

Another deviation from the terminology used here comes from Bourbaki: we do not include the Hausdorff property in the definition of compactness, while Bourbaki does. According to our definition, $\mathbb{R}_{T_1}$ is compact, but according to Bourbaki it is not.

⌈17′3⌋ Compactness in Terms of Closed Sets

A collection of subsets of a set is said to have the *finite intersection property* if each finite subcollection has a nonempty intersection.

17.F. A collection Σ of subsets of a set X has the finite intersection property iff there exists no finite $\Sigma_1 \subset \Sigma$ such that the complements of sets in Σ_1 cover X.

17.G. *A space X is compact iff every collection of closed sets in X with the finite intersection property has a nonempty intersection.*

⌈17′4⌋ Compact Sets

A *compact set* is a subset A of a topological space X (the latter must be clear from the context) provided that A is compact as a space with the relative topology induced from X.

17.H. *A subset A of a space X is compact iff each cover of A with sets open in X contains a finite subcovering.*

17.3. Is $[1, 2) \subset \mathbb{R}$ compact?

17.4. Is the same set $[1, 2)$ compact in the arrow?

17.5. Find a necessary and sufficient condition (not formulated in topological terms) for a subset of the arrow to be compact?

17.6. Prove that each subset of $\mathbb{R}_{T_1}$ is compact.

17.7. Let A and B be two compact subsets of a space X. 1) Does it follow that $A \cup B$ is compact? 2) Does it follow that $A \cap B$ is compact?

17.8. Prove that the set $A = 0 \cup \{1/n\}_{n=1}^{\infty}$ in $\mathbb{R}$ is compact.

⌈17′5⌋ Compact Sets Versus Closed Sets

17.I. Is compactness hereditary?

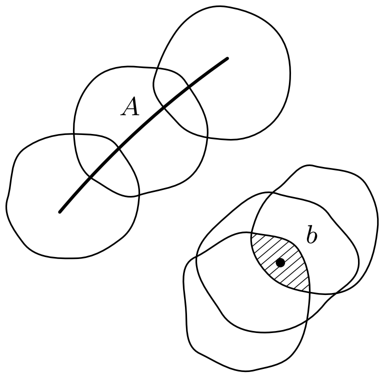
17.J. *Any closed subset of a compact space is compact.*

Theorem *17.J* can be considered a partial heredity of compactness.

In a Hausdorff space a theorem converse to *17.J* holds true:

17.K. *Any compact subset of a Hausdorff space is closed.*

The arguments proving Theorem *17.K* prove, in fact, a more detailed statement presented below. This statement is more powerful. It has direct consequences, which do not follow from the theorem.



17.L Lemma to 17.K, but not only . . . . *Let A be a compact subset of a Hausdorff space X, and let b be a point of X not in A. Then there exist open sets $U, V \subset X$ such that $b \in V$, $A \subset U$, and $U \cap V = \varnothing$.*

> ***17.9.*** Construct a nonclosed compact subset of some topological space. What is the minimal number of points needed?

⌈17′6⌋ Compactness and Separation Axioms

17.M. *A compact Hausdorff space is regular.*

17.N. Prove that a compact Hausdorff space is normal.

17.O Lemma to 17.N. Any two disjoint compact sets in a Hausdorff space possess disjoint neighborhoods.

> ***17.10.*** Prove that the intersection of any family of compact subsets of a Hausdorff space is compact. (Cf. *17.7.*)
>
> ***17.11.*** Let X be a Hausdorff space, let $\{K_\lambda\}_{\lambda\in\Lambda}$ be a family of its compact subsets, and let U be an open set containing $\bigcap_{\lambda\in\Lambda} K_\lambda$. Prove that for some finite $A \subset \Lambda$ we have $U \supset \bigcap_{\lambda\in A} K_\lambda$.
>
> ***17.12.*** Let $\{K_n\}_1^\infty$ be a decreasing sequence of nonempty compact connected sets in a Hausdorff space. Prove that the intersection $\bigcap_{n=1}^\infty K_n$ is nonempty and connected. (Cf. *12.20.*)

⌈17′7⌋ Compactness in Euclidean Space

17.P. *The segment I is compact.*

Recall that the unit n-dimensional cube (the n-cube) is the set

$$I^n = \{x \in \mathbb{R}^n \mid x_i \in [0,1] \text{ for } i = 1, \dots, n\}.$$

17.Q. *The cube I^n is compact.*

17.R. *Any compact subset of a metric space is bounded.*

Therefore, any compact subset of a metric space is closed and bounded (see Theorems *15.A*, *17.K*, and *17.R*).

17.S. Construct a closed and bounded, but noncompact set in a metric space.

17.13. Are the metric spaces of Problem *4.A* compact?

17.T. *A subset of a Euclidean space is compact iff it is closed and bounded.*

17.14. Which of the following sets are compact:

(1) $[0,1)$;	(2) ray $\mathbb{R}_+ = \{x \in \mathbb{R} \mid x \geq 0\}$;	(3) S^1;
(4) S^n;	(5) one-sheeted hyperboloid;	(6) ellipsoid;
(7) $[0,1] \cap \mathbb{Q}$?		

An $n \times k$ matrix (a_{ij}) with real entries can be regarded as a point in $\mathbb{R}^{nk}$. To do this, we only need to enumerate somehow (e.g., lexicographically) the entries of (a_{ij}) by numbers from 1 to nk. This identifies the set $L(n,k)$ of all such matrices with $\mathbb{R}^{nk}$ and endows it with a topological structure. (Cf. Section 14.)

17.15. Which of the following subsets of $L(n,n)$ are compact:

(1) $GL(n) = \{A \in L(n,n) \mid \det A \neq 0\}$;
(2) $SL(n) = \{A \in L(n,n) \mid \det A = 1\}$;
(3) $O(n) = \{A \in L(n,n) \mid A \text{ is an orthogonal matrix}\}$;
(4) $\{A \in L(n,n) \mid A^2 = \mathbb{E}\}$, where $\mathbb{E}$ is the unit matrix?

⌈17′8⌋ Compactness and Continuous Maps

17.U. *A continuous image of a compact space is compact. (In other words, if X is a compact space and $f : X \to Y$ is a continuous map, then the set $f(X)$ is compact.)*

17.V. *A continuous numerical function on a compact space is bounded and attains its maximal and minimal values. (In other words, if X is a compact space and $f : X \to \mathbb{R}$ is a continuous function, then there exist $a, b \in X$ such that $f(a) \leq f(x) \leq f(b)$ for every $x \in X$.) Cf. 17.U and 17.T.*

17.16. Prove that if $f : I \to \mathbb{R}$ is a continuous function, then $f(I)$ is a segment.

17.17. Let A be a subset of $\mathbb{R}^n$. Prove that A is compact iff each continuous numerical function on A is bounded.

17.18. Prove that if F and G are disjoint subsets of a metric space, F is closed, and G is compact, then the distance $\rho(G,F) = \inf\{\rho(x,y) \mid x \in F,\ y \in G\}$ is positive.

17.19. Prove that any open set U containing a compact set A of a metric space X contains an ***ε-neighborhood*** of A (i.e., the set $\{x \in X \mid \rho(x,A) < \varepsilon\}$) for some $\varepsilon > 0$.

17.20. Let A be a closed connected subset of $\mathbb{R}^n$, and let V be the ***closed ε-neighborhood*** of A (i.e., $V = \{x \in \mathbb{R}^n \mid \rho(x,A) \leq \varepsilon\}$). Prove that V is path-connected.

17.21. Prove that if the closure of each open ball in a compact metric space is the closed ball with the same center and radius, then any ball in this space is connected.

17.22. Let X be a compact metric space, and let $f : X \to X$ be a map such that $\rho(f(x), f(y)) < \rho(x, y)$ for any $x, y \in X$ with $x \neq y$. Prove that f has a unique fixed point. (Recall that a fixed point of f is a point x such that $f(x) = x$, see *15.6*.)

17.23. Prove that for each open cover of a compact metric space there exists a (sufficiently small) number $r > 0$ such that each open ball of radius r is contained in an element of the cover.

17.W Lebesgue Lemma. *Let $f : X \to Y$ be a continuous map from a compact metric space X to a topological space Y, and let Γ be an open cover of Y. Then there exists a number $\delta > 0$ such that for any set $A \subset X$ with diameter $\operatorname{diam}(A) < \delta$ the image $f(A)$ is contained in an element of Γ.*

⌈17′9⌋ Compactness and Closed Maps

A continuous map is *closed* if the image of each closed set under this map is closed.

17.24. A continuous bijection is a homeomorphism iff it is closed.

17.X. *A continuous map of a compact space to a Hausdorff space is closed.*

Here are two important corollaries of this theorem.

17.Y. *A continuous bijection of a compact space onto a Hausdorff space is a homeomorphism.*

17.Z. *A continuous injection of a compact space into a Hausdorff space is a topological embedding.*

17.25. Show that none of the assumptions in *17.Y* can be omitted without making the statement false.

17.26. Does there exist a noncompact subspace A of the Euclidian space such that each continuous map of A to a Hausdorff space is closed? (Cf. *17.V* and *17.X*.)

17.27. A restriction of a closed map to a closed subset is also a closed map.

17.28. Assume that $f : X \to Y$ is a continuous map, $K \subset X$ is a compact set, and Y is Hausdorff. Suppose that the restriction $f|_K$ is injective and each $a \in K$ has a neighborhood U_a such that the restriction $f|_{U_a}$ is injective. Then K has a neighborhood U such that the restriction $f|_U$ is injective.

⌈17′10x⌋ Norms in $\mathbb{R}^n$

17.29x. Prove that each norm $\mathbb{R}^n \to \mathbb{R}$ (see Section 4) is a continuous function (with respect to the standard topology of $\mathbb{R}^n$).

17.30x. Prove that any two norms in $\mathbb{R}^n$ are equivalent (i.e., determine the same topological structure). See *4.27*, cf. *4.31*.

17.31x. Does the same hold true for *metrics* on $\mathbb{R}^n$?

⌈17′11x⌋ Induction on Compactness

A function $f : X \to \mathbb{R}$ is ***locally bounded*** if for each point $a \in X$ there exist a neighborhood U and a number $M > 0$ such that $|f(x)| \le M$ for $x \in U$ (i.e., each point has a neighborhood U such that the restriction of f to U is bounded).

17.32x. Prove that if a space X is compact and a function $f : X \to \mathbb{R}$ is locally bounded, then f is bounded.

This statement is a simple application of a general principle formulated below in *17.33x*. This principle can be called ***induction on compactness*** (cf. induction on connectedness, which was discussed in Section 12).

Let X be a topological space, $\mathcal{C}$ a property of subsets of X. We say that $\mathcal{C}$ is ***additive*** if the union of each finite family of sets having the property $\mathcal{C}$ also has this property. The space X ***possesses the property $\mathcal{C}$ locally*** if each point of X has a neighborhood with property $\mathcal{C}$.

17.33x. Prove that a compact space which locally possesses an additive property has this property itself.

17.34x. Using induction on compactness, deduce the statements of Problems *17.R*, *18.M*, and *18.N*.

18. Sequential Compactness

⌈18′1⌋ Sequential Compactness Versus Compactness

A topological space is *sequentially compact* if every sequence of its points contains a convergent subsequence.

18.A. *If a first countable space is compact, then it is sequentially compact.*

A point b is an *accumulation point* of a set A if each neighborhood of b contains infinitely many points of A.

18.A.1. Prove that a point b in a space satisfying the first separation axiom is an accumulation point iff b is a limit point.

18.A.2. *Any infinite set in a compact space has an accumulation point.*

18.A.3. *A space in which each infinite set has an accumulation point is sequentially compact.*

18.B. *A sequentially compact second countable space is compact.*

18.B.1. *A decreasing sequence of nonempty closed sets in a sequentially compact space has a nonempty intersection.*

18.B.2. Prove that each nested sequence of nonempty closed sets in a space X has a nonempty intersection iff each countable collection of closed sets in X with the finite intersection property has a nonempty intersection.

18.B.3. Derive Theorem *18.B* from *18.B.1* and *18.B.2*.

18.C. *For second countable spaces, compactness and sequential compactness are equivalent.*

⌈18′2⌋ In Metric Space

A subset A of a metric space X is an ε*-net* (where ε is a positive number) if $\rho(x, A) < \varepsilon$ for each point $x \in X$.

18.D. Prove that each compact metric space contains a finite ε-net for each $\varepsilon > 0$.

18.E. Prove that each sequentially compact metric space contains a finite ε-net for each $\varepsilon > 0$.

18.F. Prove that a subset A of a metric space is everywhere dense iff A is an ε-net for each $\varepsilon > 0$.

18.G. *Any sequentially compact metric space is separable.*

18.H. Any sequentially compact metric space is second countable.

18.I. *For metric spaces, compactness and sequential compactness are equivalent.*

18.1. Prove that a sequentially compact metric space is bounded. (Cf. *18.E* and *18.I.*)

18.2. Prove that for each $\varepsilon > 0$ each metric space contains

(1) a discrete ε-net, and
(2) an ε-net such that the distance between any two of its points is greater than ε.

⌈18′3⌋ Completeness and Compactness

A sequence $\{x_n\}_{n\in\mathbb{N}}$ of points of a metric space is a *Cauchy sequence* (or a *fundamental* sequence) if for every $\varepsilon > 0$ there exists a number N such that $\rho(x_n, x_m) < \varepsilon$ for any $n, m \geq N$. A metric space X is *complete* if every Cauchy sequence in X converges.

18.J. A Cauchy sequence containing a convergent subsequence converges.

18.K. Prove that a metric space M is complete iff every nested sequence of closed balls in M with radii tending to 0 has a nonempty intersection.

18.L. Prove that a compact metric space is complete.

18.M. Prove that a complete metric space is compact iff for each $\varepsilon > 0$ it contains a finite ε-net.

18.N. Prove that a complete metric space is compact iff it contains a compact ε-net for each $\varepsilon > 0$.

⌈18′4x⌋ Noncompact Balls in Infinite Dimension

We denote by l^∞ the set of all bounded sequences of real numbers. This is a vector space with respect to the component-wise operations. There is a natural norm in it: $\|x\| = \sup\{|x_n| \mid n \in \mathbb{N}\}$.

18.3x. Are closed balls of l^∞ compact? What about spheres?

18.4x. Is the set $\{x \in l^\infty \mid |x_n| \leq 2^{-n}, n \in \mathbb{N}\}$ compact?

18.5x. Prove that the set $\{x \in l^\infty \mid |x_n| = 2^{-n}, n \in \mathbb{N}\}$ is homeomorphic to the Cantor set K introduced in Section 2.

18.6x*. Does there exist an infinitely dimensional normed space in which closed balls are compact?

⌈18′5x⌋ p-Adic Numbers

Fix a prime integer p. Denote by $\mathbb{Z}_p$ the set of series of the form $a_0 + a_1 p + \dots + a_n p^n + \dots$ with $0 \leq a_n < p$, $a_n \in \mathbb{N}$. For $x, y \in \mathbb{Z}_p$, put $\rho(x, y) = 0$ if $x = y$, and $\rho(x, y) = p^{-m}$ if m is the smallest number such that the mth coefficients in the series x and y are different.

18.7x. Prove that ρ is a metric on $\mathbb{Z}_p$.

This metric space is the ***space of integer p-adic numbers***. There is an injection $\mathbb{Z} \to \mathbb{Z}_p$ sending $a_0 + a_1 p + \dots + a_n p^n \in \mathbb{Z}$ with $0 \le a_k < p$ to the series

$$a_0 + a_1 p + \dots + a_n p^n + 0p^{n+1} + 0p^{n+2} + \dots \in \mathbb{Z}_p$$

and $-(a_0 + a_1 p + \dots + a_n p^n) \in \mathbb{Z}$ with $0 \le a_k < p$ to the series

$$b_0 + b_1 p + \dots + b_n p^n + (p-1)p^{n+1} + (p-1)p^{n+2} + \dots,$$

where

$$b_0 + b_1 p + \dots + b_n p^n = p^{n+1} - (a_0 + a_1 p + \dots + a_n p^n).$$

Cf. *4.Ux*.

18.8x. Prove that the image of the injection $\mathbb{Z} \to \mathbb{Z}_p$ is dense in $\mathbb{Z}_p$.

18.9x. Is $\mathbb{Z}_p$ a complete metric space?

18.10x. Is $\mathbb{Z}_p$ compact?

⌈18′6x⌋ Spaces of Convex Figures

Let $D \subset \mathbb{R}^2$ be a closed disk of radius p. Consider the set $\mathcal{P}_n$ of all convex polygons P with the following properties:

- the perimeter of P is at most p;
- P is contained in D;
- P has at most n vertices (the cases of one and two vertices are not excluded; the perimeter of a segment is twice its length).

See *4.Mx*, cf. *4.Ox*.

18.11x. Equip $\mathcal{P}_n$ with a natural topological structure. For instance, define a natural metric on $\mathcal{P}_n$.

18.12x. Prove that $\mathcal{P}_n$ is compact.

18.13x. Prove that $\mathcal{P}_n$ contains a polygon having the maximal area.

18.14x. Prove that this polygon is a regular n-gon.

Consider now the set $\mathcal{P}_\infty$ of all convex polygons that have perimeter at most p and are contained in D. In other words, $\mathcal{P}_\infty = \bigcup_{n=1}^{\infty} \mathcal{P}_n$.

18.15x. Construct a topological structure in $\mathcal{P}_\infty$ that induces on $\mathcal{P}_n$ the topological structures discussed above.

18.16x. Prove that the space $\mathcal{P}_\infty$ is not compact.

Consider now the set $\mathcal{P}$ of all convex closed subsets of the plane that have perimeter at most p and are contained in D. (Observe that all sets in $\mathcal{P}$ are compact.)

18.17x. Construct a topological structure in $\mathcal{P}$ that induces the structure introduced above in the space $\mathcal{P}_\infty$.

18.18x. Prove that the space $\mathcal{P}$ is compact.

18.19x. Prove that there exists a convex plane set with perimeter at most p having a maximal area.

18.20x. Prove that this is a disk of radius $p/(2\pi)$.

19x. Local Compactness and Paracompactness

⌈19′1x⌋ Local Compactness

A topological space X is *locally compact* if each point of X has a neighborhood with compact closure.

19.1x. Compact spaces are locally compact.

19.2x. Which of the following spaces are locally compact:
(1) $\mathbb{R}$; (2) $\mathbb{Q}$; (3) $\mathbb{R}^n$; (4) a discrete space?

19.3x. Find two locally compact sets on the line such that their union is not locally compact.

19.Ax. Is the local compactness hereditary?

19.Bx. A closed subset of a locally compact space is locally compact.

19.Cx. Is it true that an open subset of a locally compact space is locally compact?

19.Dx. A Hausdorff locally compact space is regular.

19.Ex. An open subset of a locally compact Hausdorff space is locally compact.

19.Fx. Local compactness is a local property for a Hausdorff space, i.e., a Hausdorff space is locally compact iff each of its points has a locally compact neighborhood.

⌈19′2x⌋ One-Point Compactification

Let (X, Ω) be a Hausdorff topological space. Let X^* be the set obtained by adding a point x_* to X (of course, x_* does not belong to X). Let Ω^* be the collection of subsets of X^* consisting of

- sets open in X and
- sets of the form $X^* \smallsetminus C$, where $C \subset X$ is a compact set:

$$\Omega^* = \Omega \cup \{X^* \smallsetminus C \mid C \subset X \text{ is a compact set}\}.$$

19.Gx. Prove that Ω^* is a topological structure on X^*.

19.Hx. Prove that the space (X^*, Ω^*) is compact.

19.Ix. Prove that the inclusion $(X, \Omega) \hookrightarrow (X^*, \Omega^*)$ is a topological embedding.

19.Jx. Prove that if X is locally compact, then the space (X^*, Ω^*) is Hausdorff. (Recall that in the definition of X^* we assumed that X is Hausdorff.)

A topological embedding of a space X in a compact space Y is a *compactification* of X if the image of X is dense in Y. In this situation, Y is also called a *compactification* of X. (To simplify the notation, we identify X with its image in Y.)

19.Kx. Prove that if X is a locally compact Hausdorff space and Y is a compactification of X with one-point complement $Y \setminus X$, then there exists a homeomorphism $Y \to X^*$ identical on X.

Any space Y of Problem *19.Kx* is called a *one-point compactification* or *Alexandrov compactification* of X. Problem *19.Kx* says that Y is essentially unique.

19.Lx. Prove that the one-point compactification of the plane is homeomorphic to S^2.

19.4x. Prove that the one-point compactification of $\mathbb{R}^n$ is homeomorphic to S^n.

19.5x. Give explicit descriptions for one-point compactifications of the following spaces:

(1) annulus $\{(x, y) \in \mathbb{R}^2 \mid 1 < x^2 + y^2 < 2\}$;
(2) square without vertices $\{(x, y) \in \mathbb{R}^2 \mid x, y \in [-1, 1], |xy| < 1\}$;
(3) strip $\{(x, y) \in \mathbb{R}^2 \mid x \in [0, 1]\}$;
(4) a compact space.

19.Mx. Prove that a locally compact Hausdorff space is regular.

19.6x. Let X be a locally compact Hausdorff space, K a compact subset of X, and U a neighborhood of K. Then K has a neighborhood V such that the closure $\operatorname{Cl} V$ is compact and contained in U.

⌈19′3x⌋ Proper Maps

A continuous map $f : X \to Y$ is *proper* if each compact subset of Y has compact preimage.

Let X and Y be two Hausdorff spaces. Any map $f : X \to Y$ obviously extends to the map

$$f^* : X^* \to Y^* : x \mapsto \begin{cases} f(x) & \text{if } x \in X, \\ y^* & \text{if } x = x^*. \end{cases}$$

19.Nx. Prove that f^* is continuous iff f is a proper continuous map.

19.Ox. Prove that each proper map of a Hausdorff space to a Hausdorff locally compact space is closed.

Problem *19.Ox* is related to Theorem *17.X*.

***19.P*x.** Extend this analogy: formulate and prove statements corresponding to Theorems *17.Z* and *17.Y*.

⌈19′4x⌋ Locally Finite Collections of Subsets

A collection Γ of subsets of a space X is *locally finite* if each point $b \in X$ has a neighborhood U that meets only finitely many sets $A \in \Gamma$.

***19.Q*x.** A locally finite cover of a compact space is finite.

***19.7*x.** If a collection Γ of subsets of a space X is locally finite, then so is $\{\operatorname{Cl} A \mid A \in \Gamma\}$.

***19.8*x.** If a collection Γ of subsets of a space X is locally finite, then each compact set $A \subset X$ meets only a finite number of sets in Γ.

***19.9*x.** If a collection Γ of subsets of a space X is locally finite and each $A \in \Gamma$ has compact closure, then each $A \in \Gamma$ meets only a finite number of sets in Γ.

***19.10*x.** Any locally finite cover of a sequentially compact space is finite.

***19.R*x.** Find an open cover of $\mathbb{R}^n$ that has no locally finite subcovering.

Let Γ and Δ be two covers of a set X. The cover Δ is a *refinement* of Γ if for each $A \in \Delta$ there exists $B \in \Gamma$ such that $A \subset B$.

***19.S*x.** Prove that any open cover of $\mathbb{R}^n$ has a locally finite open refinement.

***19.T*x.** Let $\{U_i\}_{i\in\mathbb{N}}$ be a (locally finite) open cover of $\mathbb{R}^n$. Prove that there exists an open cover $\{V_i\}_{i\in\mathbb{N}}$ of $\mathbb{R}^n$ such that $\operatorname{Cl} V_i \subset U_i$ for each $i \in \mathbb{N}$.

⌈19′5x⌋ Paracompact Spaces

A space X is *paracompact* if every open cover of X has a locally finite open refinement.

***19.U*x.** Any compact space is paracompact.

***19.V*x.** $\mathbb{R}^n$ is paracompact.

***19.W*x.** Let $X = \bigcup_{i=1}^{\infty} X_i$, where X_i are compact sets such that $X_i \subset \operatorname{Int} X_{i+1}$. Then X is paracompact.

***19.X*x.** Let X be a locally compact space. If X has a countable cover by compact sets, then X is paracompact.

***19.11*x.** Prove that if a locally compact space is second countable, then it is paracompact.

***19.12*x.** A closed subspace of a paracompact space is paracompact.

***19.13*x.** A disjoint union of paracompact spaces is paracompact.

⌈19′6x⌋ Paracompactness and Separation Axioms

***19.14*x.** Let X be a paracompact topological space, and let F and M be two disjoint subsets of X, where F is closed. Suppose that F is covered by open sets U_α whose closures are disjoint with M: $\operatorname{Cl} U_\alpha \cap M = \varnothing$. Then F and M have disjoint neighborhoods.

***19.15*x.** A Hausdorff paracompact space is regular.

***19.16*x.** A Hausdorff paracompact space is normal.

***19.17*x.** Let X be a Hausdorff locally compact and paracompact space, Γ a locally finite open cover of X. Then X has a locally finite open cover Δ such that the closures $\operatorname{Cl} V$, where $V \in \Delta$, are compact sets and $\{\operatorname{Cl} V \mid V \in \Delta\}$ is a refinement of Γ.

Here is a more general (though formally weaker) fact.

***19.18*x.** Let X be a normal space, Γ a locally finite open cover of X. Then X has a locally finite open cover Δ such that $\{\operatorname{Cl} V \mid V \in \Delta\}$ is a refinement of Γ.

Information. *Metrizable spaces are paracompact.*

⌈19′7x⌋ Partitions of Unity

Let X be a topological space, $f : X \to \mathbb{R}$ a function. Then the set $\operatorname{supp} f = \operatorname{Cl}\{x \in X \mid f(x) \neq 0\}$ is the ***support*** of f.

***19.19*x.** Let X be a topological space, and let $\{f_\alpha : X \to \mathbb{R}\}_{\alpha\in\Lambda}$ be a family of continuous functions whose supports $\operatorname{supp}(f_\alpha)$ constitute a locally finite cover of X. Prove that the formula

$$f(x) = \sum_{\alpha\in\Lambda} f_\alpha(x)$$

determines a continuous function $f : X \to \mathbb{R}$.

A family of nonnegative functions $f_\alpha : X \to \mathbb{R}_+$ is a ***partition of unity*** if the supports $\operatorname{supp}(f_\alpha)$ constitute a locally finite cover of the space X and $\sum_{\alpha\in\Lambda} f_\alpha(x) = 1$.

A partition of unity $\{f_\alpha\}$ is ***subordinate to a cover*** Γ if $\operatorname{supp}(f_\alpha)$ is contained in an element of Γ for each α. We also say that Γ ***dominates*** $\{f_\alpha\}$.

***19.Y*x.** Let X be a normal space. Then each locally finite open cover of X dominates a certain partition of unity.

***19.20*x.** Let X be a Hausdorff space. If each open cover of X dominates a certain partition of unity, then X is paracompact.

Information. *A Hausdorff space X is paracompact iff each open cover of X dominates a certain partition of unity.*

⌈19′8x⌋ Application: Making Embeddings from Pieces

***19.21*x.** Let X be a topological space, $\{U_i\}_{i=1}^k$ an open cover of X. If U_i can be embedded in $\mathbb{R}^n$ for each $i = 1, \dots, k$, then X can be embedded in $\mathbb{R}^{k(n+1)}$.

> ***19.21*x*.1.*** Let $h_i : U_i \to \mathbb{R}^n$, $i = 1, \dots, k$, be embeddings, and let $f_i : X \to \mathbb{R}$ form a partition of unity subordinate to the cover $\{U_i\}_{i=1}^k$. We put $\hat{h}_i(x) = (h_i(x), 1) \in \mathbb{R}^{n+1}$. Show that the map $X \to \mathbb{R}^{k(n+1)} : x \mapsto (f_i(x)\hat{h}_i(x))_{i=1}^k$ is an embedding.

19.22*x. *Riddle. How can you generalize *19.21x*?

Chapter IV

Topological Constructions

20. Multiplication

⌈20′1⌋ Set-Theoretic Digression: Product of Sets

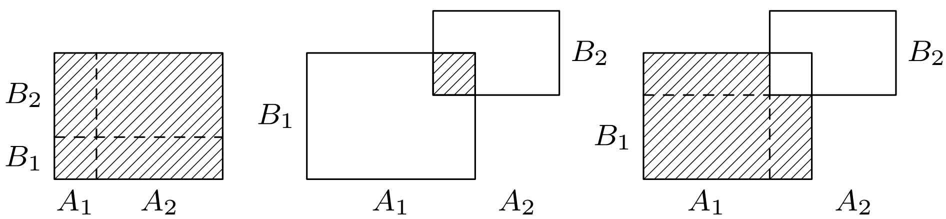
Let X and Y be two sets. The set of ordered pairs (x, y) with $x \in X$ and $y \in Y$ is called the *direct product*, *Cartesian product*, or just *product* of X and Y and denoted by $X \times Y$. If $A \subset X$ and $B \subset Y$, then $A \times B \subset X \times Y$. Sets $X \times b$ with $b \in Y$ and $a \times Y$ with $a \in X$ are *fibers* of the product $X \times Y$.

20.A. Prove that for any $A_1, A_2 \subset X$ and $B_1, B_2 \subset Y$ we have

$$(A_1 \cup A_2) \times (B_1 \cup B_2) = (A_1 \times B_1) \cup (A_1 \times B_2) \cup (A_2 \times B_1) \cup (A_2 \times B_2),$$

$$(A_1 \times B_1) \cap (A_2 \times B_2) = (A_1 \cap A_2) \times (B_1 \cap B_2),$$

$$(A_1 \times B_1) \smallsetminus (A_2 \times B_2) = \big((A_1 \smallsetminus A_2) \times B_1\big) \cup \big(A_1 \times (B_1 \smallsetminus B_2)\big).$$



The natural maps

$$\mathrm{pr}_X : X \times Y \to X : (x, y) \mapsto x \quad \text{and} \quad \mathrm{pr}_Y : X \times Y \to Y : (x, y) \mapsto y$$

are (*natural*) *projections*.

20.B. Prove that $\operatorname{pr}_X^{-1}(A) = A \times Y$ for each $A \subset X$.

> ***20.1.*** Find the corresponding formula for $B \subset Y$.

⌈20′2⌋ Graphs

A map $f : X \to Y$ determines a subset Γ_f of $X \times Y$ defined by $\Gamma_f = \{(x, f(x)) \mid x \in X\}$, it is called the *graph* of f.

20.C. A set $\Gamma \subset X \times Y$ is the graph of a map $X \to Y$ iff for each $a \in X$ the intersection $\Gamma \cap (a \times Y)$ is a singleton.

> ***20.2.*** Prove that for each map $f : X \to Y$ and each set $A \subset X$ we have
> $$f(A) = \operatorname{pr}_Y(\Gamma_f \cap (A \times Y)) = \operatorname{pr}_Y(\Gamma_f \cap \operatorname{pr}_X^{-1}(A))$$
> and $f^{-1}(B) = \operatorname{pr}_X(\Gamma \cap (X \times B))$ for each $B \subset Y$.
>
> The set $\Delta = \{(x, x) \mid x \in X\} = \{(x, y) \in X \times X \mid x = y\}$ is the *diagonal* of $X \times X$.
>
> ***20.3.*** Let A and B be two subsets of X. Prove that $(A \times B) \cap \Delta = \varnothing$ iff $A \cap B = \varnothing$.
>
> ***20.4.*** Prove that the map $\operatorname{pr}_X\big|_{\Gamma_f}$ is bijective.
>
> ***20.5.*** Prove that f is injective iff $\operatorname{pr}_Y\big|_{\Gamma_f}$ is injective.
>
> ***20.6.*** Consider the map $T : X \times Y \to Y \times X : (x, y) \mapsto (y, x)$. Prove that $\Gamma_{f^{-1}} = T(\Gamma_f)$ for each invertible map $f : X \to Y$.

⌈20′3⌋ Product of Topologies

Let X and Y be two topological spaces. If U is an open set of X and B is an open set of Y, then we say that $U \times V$ is an *elementary open set* of $X \times Y$.

20.D. *The set of elementary open sets of $X \times Y$ is a base of a topological structure in $X \times Y$.*

The topological structure determined by the base of elementary open sets is *the product topology* in $X \times Y$. The *product* of two spaces X and Y is the set $X \times Y$ with the product topology.

> ***20.7.*** Prove that for any subspaces A and B of spaces X and Y the product topology on $A \times B$ coincides with the topology induced from $X \times Y$ via the natural inclusion $A \times B \subset X \times Y$.

20.E. $Y \times X$ is canonically homeomorphic to $X \times Y$.

The word *canonically* here means that the homeomorphism between $X \times Y$ and $Y \times X$, which exists according to the statement, can be chosen in a nice special (or even obvious?) way, and so we may expect that it has additional pleasant properties.

20.F. The canonical bijection $X \times (Y \times Z) \to (X \times Y) \times Z$ is a homeomorphism.

20.8. Prove that if A is closed in X and B is closed in Y, then $A \times B$ is closed in $X \times Y$.

20.9. Prove that $\operatorname{Cl}(A \times B) = \operatorname{Cl} A \times \operatorname{Cl} B$ for any $A \subset X$ and $B \subset Y$.

20.10. Is it true that $\operatorname{Int}(A \times B) = \operatorname{Int} A \times \operatorname{Int} B$?

20.11. Is it true that $\operatorname{Fr}(A \times B) = \operatorname{Fr} A \times \operatorname{Fr} B$?

20.12. Is it true that $\operatorname{Fr}(A \times B) = (\operatorname{Fr} A \times B) \cup (A \times \operatorname{Fr} B)$?

20.13. Prove that $\operatorname{Fr}(A \times B) = (\operatorname{Fr} A \times B) \cup (A \times \operatorname{Fr} B)$ for closed A and B.

20.14. Find a formula for $\operatorname{Fr}(A \times B)$ in terms of A, $\operatorname{Fr} A$, B, and $\operatorname{Fr} B$.

⌈20′4⌋ Topological Properties of Projections and Fibers

20.G. *The natural projections* $\operatorname{pr}_X : X \times Y \to X$ *and* $\operatorname{pr}_Y : X \times Y \to Y$ *are continuous for any topological spaces* X *and* Y*.*

20.H. The product topology is the coarsest topology with respect to which pr_X and pr_Y are continuous.

20.I. *A fiber of a product is canonically homeomorphic to the corresponding factor. The canonical homeomorphism is the restriction to the fiber of the natural projection of the product onto the factor.*

20.J. Prove that $\mathbb{R}^1 \times \mathbb{R}^1 = \mathbb{R}^2$, $(\mathbb{R}^1)^n = \mathbb{R}^n$, and $(I)^n = I^n$. (We remind the reader that I^n is the n-dimensional unit cube in $\mathbb{R}^n$.)

20.15. Let Σ_X and Σ_Y be bases of spaces X and Y. Prove that the sets $U \times V$ with $U \in \Sigma_X$ and $V \in \Sigma_Y$ constitute a base for $X \times Y$.

20.16. Prove that a map $f : X \to Y$ is continuous iff $\operatorname{pr}_X|_{\Gamma_f} : \Gamma_f \to X$ is a homeomorphism.

20.17. Prove that if W is open in $X \times Y$, then $\operatorname{pr}_X(W)$ is open in X.

A map from a space X to a space Y is *open* (*closed*) if the image of each open set under this map is open (respectively, closed). Therefore, *20.17* states that $\operatorname{pr}_X : X \times Y \to X$ is an open map.

20.18. Is pr_X a closed map?

20.19. Prove that for each space X and each compact space Y the map $\operatorname{pr}_X : X \times Y \to X$ is closed.

⌈20′5⌋ Cartesian Products of Maps

Let X, Y, and Z be three sets. A map $f : Z \to X \times Y$ determines the compositions $f_1 = \operatorname{pr}_X \circ f : Z \to X$ and $f_2 = \operatorname{pr}_Y \circ f : Z \to Y$, which are called the *factors* (or *components*) of f. Indeed, f is determined by them as a sort of product.

20.K. Prove that for any maps $f_1 : Z \to X$ and $f_2 : Z \to Y$ there exists a unique map $f : Z \to X \times Y$ with $\mathrm{pr}_X \circ f = f_1$ and $\mathrm{pr}_Y \circ f = f_2$.

20.20. Prove that $f^{-1}(A \times B) = f_1^{-1}(A) \cap f_2^{-1}(B)$ for any $A \subset X$ and $B \subset Y$.

20.L. Let X, Y, and Z be three spaces. Prove that $f : Z \to X \times Y$ is continuous iff so are f_1 and f_2.

Any two maps $g_1 : X_1 \to Y_1$ and $g_2 : X_2 \to Y_2$ determine a map

$$g_1 \times g_2 : X_1 \times X_2 \to Y_1 \times Y_2 : (x_1, x_2) \mapsto (g_1(x_1), g_2(x_2)),$$

which is their (*Cartesian*) *product*.

20.21. Prove that $(g_1 \times g_2)(A_1 \times A_2) = g_1(A_1) \times g_2(A_2)$ for any $A_1 \subset X_1$ and $A_2 \subset X_2$.

20.22. Prove that $(g_1 \times g_2)^{-1}(B_1 \times B_2) = g_1^{-1}(B_1) \times g_2^{-1}(B_2)$ for any $B_1 \subset Y_1$ and $B_2 \subset Y_2$.

20.M. Prove that the Cartesian product of continuous maps is continuous.

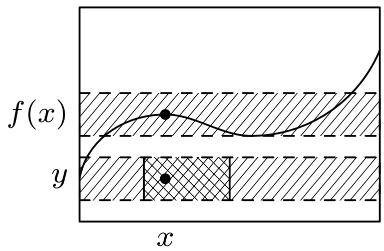
20.23. Prove that the Cartesian product of open maps is open.

20.24. Prove that a metric $\rho : X \times X \to \mathbb{R}$ is continuous with respect to the metric topology.

20.25. Let $f : X \to Y$ be a map. Prove that the graph Γ_f is the preimage of the diagonal $\Delta_Y = \{(y, y) \mid y \in Y\} \subset Y \times Y$ under the map $f \times \mathrm{id}_Y : X \times Y \to Y \times Y$.

⌈20′6⌋ Properties of Diagonal and Other Graphs

20.26. Prove that a space X is Hausdorff iff the diagonal $\Delta = \{(x, x) \mid x \in X\}$ is closed in $X \times X$.



20.27. Prove that if Y is a Hausdorff space and $f : X \to Y$ is a continuous map, then the graph Γ_f is closed in $X \times Y$.

20.28. Let Y be a compact space. Prove that if a map $f : X \to Y$ has closed graph Γ_f, then f is continuous.

20.29. Prove that the hypothesis on compactness in *20.28* is necessary.

20.30. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. Prove that its graph is:

(1) closed;
(2) connected;
(3) path-connected;

(4) locally connected;
(5) locally compact.

20.31. Consider the following functions
1) $\mathbb{R} \to \mathbb{R} : x \mapsto \begin{cases} 0 & \text{if } x = 0, \\ 1/x, & \text{otherwise.} \end{cases}$; 2) $\mathbb{R} \to \mathbb{R} : x \mapsto \begin{cases} 0 & \text{if } x = 0, \\ \sin(1/x), & \text{otherwise.} \end{cases}$ Do their graphs possess the properties listed in *20.30*?

20.32. Does any of the properties of the graph of a function f that are mentioned in *20.30* imply that f is continuous?

20.33. Let Γ_f be closed. Then the following assertions are equivalent:

(1) f is continuous;
(2) f is locally bounded;
(3) the graph Γ_f of f is connected;
(4) the graph Γ_f of f is path-connected.

20.34. Prove that if Γ_f is connected and locally connected, then f is continuous.

20.35. Prove that if Γ_f is connected and locally compact, then f is continuous.

20.36. Are some of the assertions in Problems *20.33–20.35* true for maps $f : \mathbb{R}^2 \to \mathbb{R}$?

⌈20′7⌋ Topological Properties of Products

20.N. *The product of Hausdorff spaces is Hausdorff.*

20.37. Prove that the product of regular spaces is regular.

20.38. The product of normal spaces is not necessarily normal.

20.38.1*. Prove that the space $\mathcal{R}$ formed by real numbers with the topology determined by the base consisting of all semi-open intervals $[a, b)$ is normal.

20.38.2. Prove that in the Cartesian square of the space introduced in *20.38.1* the subspace $\{(x, y) \mid x = -y\}$ is closed and discrete.

20.38.3. Find two disjoint subsets of $\{(x, y) \mid x = -y\}$ that have no disjoint neighborhoods in the Cartesian square of the space of *20.38.1*.

20.O. *The product of separable spaces is separable.*

20.P. *First countability of factors implies first countability of the product.*

20.Q. *The product of second countable spaces is second countable.*

20.R. *The product of metrizable spaces is metrizable.*

20.S. *The product of connected spaces is connected.*

20.39. Prove that for connected spaces X and Y and any proper subsets $A \subset X$ and $B \subset Y$ the set $X \times Y \smallsetminus A \times B$ is connected.

20.T. *The product of path-connected spaces is path-connected.*

20.U. *The product of compact spaces is compact.*

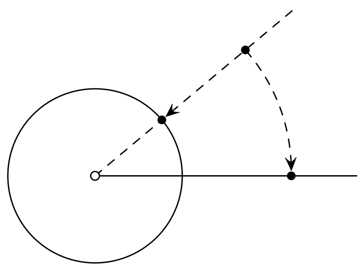
20.40. Prove that the product of locally compact spaces is locally compact.

20.41. If X is a paracompact space and Y is compact, then $X \times Y$ is paracompact.

20.42. For which of the topological properties studied above is it true that if $X \times Y$ possesses the property, then so does X?

⌈20′8⌋ Representation of Special Spaces as Products

20.V. Prove that $\mathbb{R}^2 \setminus 0$ is homeomorphic to $S^1 \times \mathbb{R}$.



20.43. Prove that $\mathbb{R}^n \setminus \mathbb{R}^k$ is homeomorphic to $S^{n-k-1} \times \mathbb{R}^{k+1}$.

20.44. Prove that $S^n \cap \{x \in \mathbb{R}^{n+1} \mid x_1^2 + \dots + x_k^2 \leq x_{k+1}^2 + \dots + x_{n+1}^2\}$ is homeomorphic to $S^{k-1} \times D^{n-k+1}$.

20.45. Prove that $O(n)$ is homeomorphic to $SO(n) \times O(1)$.

20.46. Prove that $GL(n)$ is homeomorphic to $SL(n) \times GL(1)$.

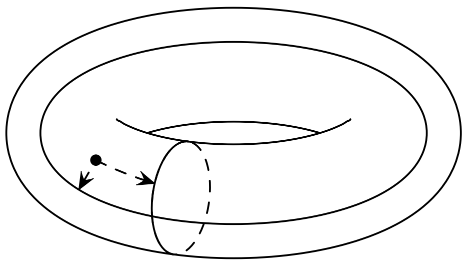
20.47. Prove that $GL_+(n)$ is homeomorphic to $SO(n) \times \mathbb{R}^{n(n+1)/2}$, where

$$GL_+(n) = \{A \in L(n,n) \mid \det A > 0\}.$$

20.48. Prove that $SO(4)$ is homeomorphic to $S^3 \times SO(3)$.

The space $S^1 \times S^1$ is a *torus*.

20.W. Construct a topological embedding of the torus in $\mathbb{R}^3$.



The product $S^1 \times \dots \times S^1$ of k factors is the *k-dimensional torus*.

20.X. Prove that the k-dimensional torus can be topologically embedded in $\mathbb{R}^{k+1}$.

20.Y. Find topological embeddings of $S^1 \times D^2$, $S^1 \times S^1 \times I$, and $S^2 \times I$ in $\mathbb{R}^3$.

21. Quotient Spaces

⌈21′1⌋ Set-Theoretic Digression: Partitions and Equivalence Relations

Recall that a *partition* of a set A is a cover of A consisting of pairwise disjoint sets.

Each partition of a set X determines an *equivalence relation* (i.e., a relation, which is reflexive, symmetric, and transitive): two elements of X are said to be equivalent if they belong to the same element of the partition. Vice versa, each equivalence relation on X determines the partition of X into classes of equivalent elements. Thus, partitions of a set into nonempty subsets and equivalence relations on the set are essentially the same. More precisely, they are two ways of describing the same phenomenon.

Let X be a set, S a partition of X. The set whose elements are members of the partition S (which are subsets of X) is the *quotient set* or *factor set* of X by S. It is denoted by $X/_S$. [1]

21.1. Riddle. How does this operation relate to division of numbers? Why is there a similarity in terminology and notation?

The set $X/_S$ is also called the *set of equivalence classes* for the equivalence relation corresponding to the partition S.

The map $\mathrm{pr} : X \to X/_S$ that sends $x \in X$ to the element of S containing x is the (*canonical*) *projection* or *factorization map*. A subset of X which is a union of elements of a partition is *saturated*. The smallest saturated set containing a subset A of X is the *saturation* of A.

21.2. Prove that $A \subset X$ is an element of a partition S of X iff $A = \mathrm{pr}^{-1}(\text{point})$, where $\mathrm{pr} : X \to X/_S$ is the natural projection.

21.A. Prove that the saturation of a set A equals $\mathrm{pr}^{-1}\big(\mathrm{pr}(A)\big)$.

21.B. Prove that a set is saturated iff it is equal to its saturation.

[1]At first glance, the definition of a quotient set contradicts one of the very profound principles of the set theory, which states that a set is determined by its elements. Indeed, according to this principle, we have $X/_S = S$ since S and $X/_S$ have the same elements. Hence, there seems to be no need to introduce $X/_S$. The real sense of the notion of a quotient set lies not in its literal set-theoretic meaning, but in our way of thinking about elements of partitions. If we remember that they are subsets of the original set and want to keep track of their internal structure (or, at least, of their elements), then we speak of a partition. If we think of them as atoms, getting rid of their possible internal structure, then we speak about the quotient set.

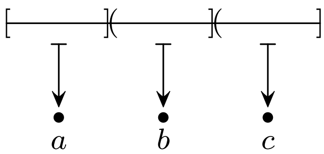
⌈21′2⌋ Quotient Topology

A quotient set $X/_S$ of a topological space X with respect to a partition S into nonempty subsets is equipped with a natural topology: a set $U \subset X/_S$ is said to be open in $X/_S$ if its preimage $\mathrm{pr}^{-1}(U)$ under the canonical projection $\mathrm{pr} : X \to X/_S$ is open.

21.C. *The collection of these sets is a topological structure in the quotient set* $X/_S$.

This topological structure is the ***quotient topology***. The set $X/_S$ with this topology is the ***quotient space*** of X by partition S.

> ***21.3.*** Give an explicit description of the quotient space of the segment $[0, 1]$ by the partition consisting of $[0, 1/3]$, $(1/3, 2/3]$, and $(2/3, 1]$.



> ***21.4.*** What can you say about a partition S of a space X if the quotient space $X/_S$ is known to be discrete?

21.D. *A subset of a quotient space* $X/_S$ *is open iff it is the image of an open saturated set under the canonical projection* pr.

21.E. A subset of a quotient space $X/_S$ is closed, iff its preimage under pr is closed in X, iff it is the image of a closed saturated set.

21.F. *The canonical projection* $\mathrm{pr} : X \to X/_S$ *is continuous.*

21.G. Prove that the quotient topology is the finest topology on $X/_S$ such that the canonical projection pr is continuous with respect to it.

⌈21′3⌋ Topological Properties of Quotient Spaces

21.H. *A quotient space of a connected space is connected.*

21.I. *A quotient space of a path-connected space is path-connected.*

21.J. *A quotient space of a separable space is separable.*

21.K. *A quotient space of a compact space is compact.*

21.L. The quotient space of the real line by the partition $\mathbb{R}_+$, $\mathbb{R} \setminus \mathbb{R}_+$ is not Hausdorff.

21.M. The quotient space of a space X by a partition S is Hausdorff iff any two elements of S have disjoint saturated neighborhoods.

21.5. Formulate similar necessary and sufficient conditions for a quotient space to satisfy other separation axioms and countability axioms.

21.6. Give an example showing that the second countability can get lost when we pass to a quotient space.

⌈21′4⌋ Set-Theoretic Digression: Quotients and Maps

Let S be a partition of a set X into nonempty subsets. Let $f : X \to Y$ be a map which is constant on each element of S. Then there is a map $X/_S \to Y$ which sends each element A of S to the element $f(a)$, where $a \in A$. This map is denoted by $f/_S$ and called the *quotient map* or *factor map* of f (by the partition S).

21.N. 1) Prove that a map $f : X \to Y$ is constant on each element of a partition S of X iff there exists a map $g : X/_S \to Y$ such that the following diagram is commutative:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\mathrm{pr}"'] & Y \\ X/_S \arrow[ur, "g"'] & \end{tikzcd}$$

2) Prove that such a map g coincides with $f/_S$.

More generally, let S and T be partitions of sets X and Y. Then every map $f : X \to Y$ that maps each subset in S to a subset in T determines a map $X/_S \to Y/_T$ which sends an element A of the partition S to the element of the partition T containing $f(A)$. This map is denoted by $f/_{(S,T)}$ and called the *quotient map* or *factor map* of f (*with respect to* S and T).

21.O. Formulate and prove for $f/_{S,T}$ a statement generalizing *21.N.*

A map $f : X \to Y$ determines the partition of the set X into nonempty preimages of the elements of Y. This partition is denoted by $S(f)$.

21.P. The map $f/_{S(f)} : X/_{S(f)} \to Y$ is injective.

This map is the *injective factor* (or *injective quotient*) of f.

⌈21′5⌋ Continuity of Quotient Maps

21.Q. *Let X and Y be two spaces, S a partition of X into nonempty sets, and $f : X \to Y$ a continuous map constant on each element of S. Then the factor $f/_S$ of f is continuous.*

21.7. If the map f is open, then so is the quotient map $f/_S$.

21.8. Let X and Y be two spaces, S a partition of X into nonempty sets. Prove that the formula $f \mapsto f/_S$ determines a bijection from the set of all continuous maps $X \to Y$ that are constant on each element of S onto the set of all continuous maps $X/_S \to Y$.

21.R. Let X and Y be two spaces, let S and T be partitions of X and Y, respectively, and let $f : X \to Y$ be a continuous map that maps each set in S to a set in T. Then the map $f/_{S,T} : X/_S \to Y/_T$ is continuous.

⌈21′6x⌋ Closed Partitions

A partition S of a space X is *closed* if the saturation of each closed set is closed.

***21.9*x.** Prove that a partition is closed iff the canonical projection $X \to X/_S$ is a closed map.

***21.10*x.** Prove that if a partition S contains only one element consisting of more than one point, then S is closed if this element is a closed set.

***21.S*x.** Let X be a space satisfying the first separation axiom, S a closed partition of X. Then the quotient space $X/_S$ also satisfies the first separation axiom.

***21.T*x.** *The quotient space of a normal space with respect to a closed partition is normal.*

⌈21′7x⌋ Open Partitions

A partition S of a space X is *open* if the saturation of each open set is open.

***21.11*x.** Prove that a partition S is open iff the canonical projection $X \to X/_S$ is an open map.

***21.12*x.** Prove that if a set A is saturated with respect to an open partition, then $\operatorname{Int} A$ and $\operatorname{Cl} A$ are also saturated.

***21.U*x.** *The quotient space of a second countable space with respect to an open partition is second countable.*

***21.V*x.** *The quotient space of a first countable space with respect to an open partition is first countable.*

***21.W*x.** *Let X and Y be two spaces, S and T their open partitions. Denote by $S \times T$ the partition of $X \times Y$ consisting of $A \times B$ with $A \in S$ and $B \in T$. Then the injective factor $X \times Y/_{S \times T} \to X/_S \times Y/_T$ of $\mathrm{pr}_S \times \mathrm{pr}_T : X \times Y \to X/_S \times Y/_T$ is a homeomorphism.*

22. Zoo of Quotient Spaces

⌈22′1⌋ Tool for Identifying a Quotient Space with a Known Space

22.A. *If X is a compact space, Y is a Hausdorff space, and $f : X \to Y$ is a continuous map, then the injective factor $f/_{S(f)} : X/_{S(f)} \to Y$ is a homeomorphism.*

22.B. *The injective factor of a continuous map from a compact space to a Hausdorff one is a topological embedding.*

> ***22.1.*** Describe explicitly partitions of a segment such that the corresponding quotient spaces are all letters of the alphabet.
>
> ***22.2.*** Prove that the segment I admits a partition with the quotient space homeomorphic to square $I \times I$.

⌈22′2⌋ Tools for Describing Partitions

An accurate literal description of a partition can often be somewhat cumbersome, but usually it can be shortened and made more understandable. Certainly, this requires a more flexible vocabulary with lots of words having almost the same meanings. For instance, such words as *factorize* and *pass to a quotient* can be replaced by *attach*, *glue together*, *identify*, *contract*, *paste*, and other words substituting or accompanying these in everyday life.

Some elements of this language are easy to formalize. For instance, factorization of a space X with respect to a partition consisting of a set A and singletons in the complement of A is the *contraction* (of the subset A to a point), and the result is denoted by $X/_A$.

> ***22.3.*** Let $A, B \subset X$ form a fundamental cover of a space X. Prove that the quotient map $A/_{A \cap B} \to X/_B$ of the inclusion $A \hookrightarrow X$ is a homeomorphism.

If A and B are two disjoint subspaces of a space X and $f : A \to B$ is a homeomorphism, then passing to the quotient of X by the partition into singletons in $X \smallsetminus (A \cup B)$ and two-element sets $\{x, f(x)\}$, where $x \in A$, we *glue* or *identify* the sets A and B via the homeomorphism f.

A rather convenient and flexible way for describing partitions is to describe the corresponding equivalence relations. The main advantage of this approach is that, by transitivity, it suffices to specify only some pairs of equivalent elements: if one states that $x \sim y$ and $y \sim z$, then it is not necessary to state that $x \sim z$ since this is automatically true.

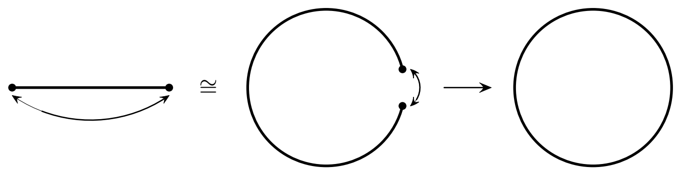
Hence, a partition is represented by a list of statements of the form $x \sim y$ that are sufficient for recovering the equivalence relation. We denote the corresponding partition by such a list enclosed into square brackets. For example, the quotient of a space X obtained by identifying subsets A and B by a homeomorphism $f : A \to B$ is denoted by $X/_{[a \sim f(a) \text{ for any } a \in A]}$ or just $X/_{[a \sim f(a)]}$.

Some partitions are easily described by a picture, especially if the original space can be embedded in the plane. In such a case, as in the pictures below, we draw arrows on the segments to be identified to show the directions to be identified.

Below we introduce all kinds of descriptions for partitions and give examples of their usage, simultaneously providing literal descriptions. The latter are not that nice, but they may help the reader to remain confident about the meaning of the new words. On the other hand, the reader will appreciate the improvement the new words bring in.

⌈22′3⌋ Welcome to the Zoo

22.C. Prove that $I/_{[0 \sim 1]}$ is homeomorphic to S^1.



In other words, the quotient space of segment I by the partition consisting of $\{0, 1\}$ and $\{a\}$ with $a \in (0, 1)$ is homeomorphic to a circle.

22.C.1. Find a surjective continuous map $I \to S^1$ such that the corresponding partition into preimages of points consists of singletons in the interior of the segment and the pair of boundary points of the segment.

22.D. Prove that $D^n/_{S^{n-1}}$ is homeomorphic to S^n.

In *22.D*, we deal with the quotient space of the n-disk D^n by the partition $\{S^{n-1}\} \cup \{\{x\} \mid x \in B^n\}$.

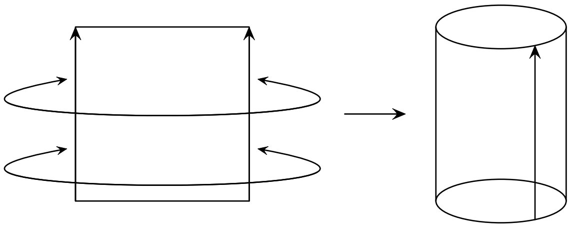
Here is a reformulation of *22.D*: *Contracting* the boundary of an n-dimensional ball to a point, we obtain an n-dimensional sphere.

22.D.1. Find a continuous map of the n-disk D^n to the n-sphere S^n that maps the boundary of the disk to a single point and bijectively maps the interior of the disk onto the complement of this point.

22.E. Prove that $I^2/_{[(0,t) \sim (1,t) \text{ for } t \in I]}$ is homeomorphic to $S^1 \times I$.

Here the partition consists of pairs of points $\{(0, t), (1, t)\}$ where $t \in I$, and singletons in $(0, 1) \times I$.

Reformulation of *22.E*: If we *glue* the side edges of a square by identifying points on the same hight, then we obtain a cylinder.



22.F. $S^1 \times I/_{[(z,0) \sim (z,1) \text{ for } z \in S^1]}$ is homeomorphic to $S^1 \times S^1$.

Here the partition consists of singletons in $S^1 \times (0,1)$ and pairs of points of the basis circles lying on the same element of the cylinder.

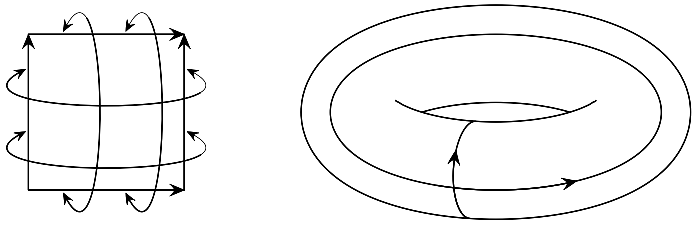
Here is a reformulation of *22.F*: If we *glue* the base circles of a cylinder by identifying pairs of points on the same element, then we obtain a torus.

22.G. $I^2/_{[(0,t) \sim (1,t),\ (t,0) \sim (t,1)]}$ is homeomorphic to $S^1 \times S^1$.

In *22.G*, the partition consists of

- singletons in the interior $(0,1) \times (0,1)$ of the square,
- pairs of points on the vertical sides that are the same distance from the bottom side (i.e., pairs $\{(0,t),(1,t)\}$ with $t \in (0,1)$),
- pairs of points on the horizontal sides that lie on the same vertical line (i.e., pairs $\{(t,0),(t,1)\}$ with $t \in (0,1)$),
- the four vertices of the square.

Reformulation of *22.G*: Identifying the sides of a square according to the picture, we obtain a torus.



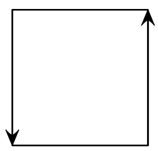
⌈22′4⌋ Transitivity of Factorization

A solution of Problem *22.G* can be based on Problems *22.E* and *22.F* and the following general theorem.

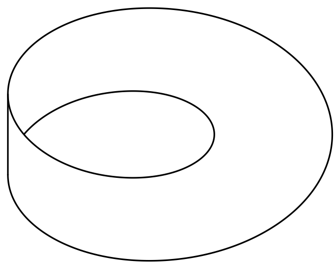
22.H Transitivity of Factorization. *Let S be a partition of a space X, and let S' be a partition of the space $X/_S$. Then the quotient space $(X/_S)/_{S'}$ is canonically homeomorphic to $X/_T$, where T is the partition of X into preimages of elements of S' under the projection $X \to X/_S$.*

⌈22′5⌋ Möbius Strip

The *Möbius strip* or *Möbius band* is defined as $I^2/_{[(0,t)\sim(1,1-t)]}$. In other words, this is the quotient space of the square I^2 by the partition into centrally symmetric pairs of points on the vertical edges of I^2, and singletons that do not lie on the vertical edges. The Möbius strip is obtained, so to speak, by identifying the vertical sides of a square in such a way that the directions shown on them by arrows are superimposed, as shown below.



22.I. Prove that the Möbius strip is homeomorphic to the surface that is swept in $\mathbb{R}^3$ by a segment rotating in a half-plane around the midpoint, while the half-plane rotates around its boundary line. The ratio of the angular velocities of these rotations is such that the rotation of the half-plane through 360° takes the same time as the rotation of the segment through 180°. See below.



⌈22′6⌋ Contracting Subsets

22.4. Prove that $[0,1]/_{[1/3,2/3]}$ is homeomorphic to $[0,1]$, and $[0,1]/_{\{1/3,1\}}$ is homeomorphic to letter P.

22.5. Prove that the following spaces are homeomorphic:
(1) $\mathbb{R}^2$; (2) $\mathbb{R}^2/_I$; (3) $\mathbb{R}^2/_{D^2}$; (4) $\mathbb{R}^2/_{I^2}$;
(5) $\mathbb{R}^2/_A$, where A is the union of several segments with a common end point;
(6) $\mathbb{R}^2/_B$, where B is a simple polyline, i.e., the union of a finite sequence of segments $I_1, \dots, I_n$ such that the initial point of I_{i+1} is the final point of I_i.

22.6. Prove that if $f : X \to Y$ is a homeomorphism, then the quotient spaces $X/_A$ and $Y/_{f(A)}$ are homeomorphic.

22.7. Let $A \subset \mathbb{R}^2$ be the ray $\{(x,y) \mid x \geq 0, y = 0\}$. Is $\mathbb{R}^2/_A$ homeomorphic to $\operatorname{Int} D^2 \cup \{(0,1)\}$?

⌈22′7⌋ Further Examples

22.8. Prove that $S^1/_{[z \sim e^{2\pi i/3} z]}$ is homeomorphic to S^1.

The partition in *22.8* consists of triples of points that are vertices of equilateral inscribed triangles.

22.9. Prove that the following quotient spaces of the disk D^2 are homeomorphic to D^2:

(1) $D^2/_{[(x, y) \sim (-x, -y)]}$,
(2) $D^2/_{[(x, y) \sim (x, -y)]}$,
(3) $D^2/_{[(x, y) \sim (-y, x)]}$.

22.10. Find a generalization of *22.9* with D^n substituted for D^2.

22.11. Describe explicitly the quotient space of the line $\mathbb{R}^1$ by the equivalence relation $x \sim y \Leftrightarrow x - y \in \mathbb{Z}$.

22.12. Represent the Möbius strip as a quotient space of cylinder $S^1 \times I$.

⌈22′8⌋ Klein Bottle

The *Klein bottle* is $I^2/_{[(t, 0) \sim (t, 1),\ (0, t) \sim (1, 1 - t)]}$. In other words, this is the quotient space of square I^2 by the partition into

- singletons in its interior,
- pairs of points $(t, 0), (t, 1)$ on horizontal edges that lie on the same vertical line,
- pairs of points $(0, t), (1, 1-t)$ symmetric with respect to the center of the square that lie on the vertical edges, and
- the quadruple of vertices.

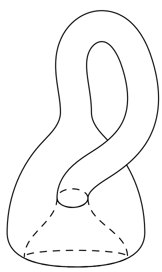
22.13. Present the Klein bottle as a quotient space of

(1) a cylinder;
(2) the Möbius strip.

22.14. Prove that $S^1 \times S^1/_{[(z, w) \sim (-z, \bar{w})]}$ is homeomorphic to the Klein bottle. (Here $\bar{w}$ denotes the complex number conjugate to w.)

22.15. Embed the Klein bottle in $\mathbb{R}^4$ (cf. *22.I* and *20.W*).

22.16. Embed the Klein bottle in $\mathbb{R}^4$ so that the image of this embedding under the orthogonal projection $\mathbb{R}^4 \to \mathbb{R}^3$ would look as follows:



⌈22′9⌋ Projective Plane

Let us identify each boundary point of the disk D^2 with the antipodal point, i.e., we factorize the disk by the partition consisting of singletons in the interior of the disk and pairs of points on the boundary circle symmetric with respect to the center of the disk. The result is the *projective plane*. This space cannot be embedded in $\mathbb{R}^3$, too. Thus, we are not able to draw it. Instead, we present it differently.

22.J. A projective plane is a result of gluing together a disk and a Möbius strip via a homeomorphism between their boundary circles.

⌈22′10⌋ You May Have Been Provoked to Perform an Illegal Operation

Solving the previous problem, you did something that did not fit into the theory presented above. Indeed, the operation with two spaces called *gluing* in *22.J* has not appeared yet. It is a combination of two operations: first, we make a single space consisting of disjoint copies of the original spaces, and then we factorize this space by identifying points of one copy with points of another. Let us consider the first operation in detail.

⌈22′11⌋ Set-Theoretic Digression: Sums of Sets

The (*disjoint*) *sum* of a family of sets $\{X_\alpha\}_{\alpha\in A}$ is the set of pairs (x_α, α) such that $x_\alpha \in X_\alpha$. The sum is denoted by $\bigsqcup_{\alpha\in A} X_\alpha$. So, we can write

$$\bigsqcup_{\alpha\in A} X_\alpha = \bigcup_{\alpha\in A} (X_\alpha \times \{\alpha\}).$$

For each $\beta \in A$, we have a natural injection

$$\mathrm{in}_\beta : X_\beta \to \bigsqcup_{\alpha\in A} X_\alpha : x \mapsto (x, \beta).$$

If only two sets X and Y are involved and they are distinct, then we can avoid indices and define the sum by setting

$$X \sqcup Y = \{(x, X) \mid x \in X\} \cup \{(y, Y) \mid y \in Y\}.$$

⌈22′12⌋ Sums of Spaces

22.K. Let $\{X_\alpha\}_{\alpha\in A}$ be a collection of topological spaces. Then the collection of subsets of $\bigsqcup_{\alpha\in A} X_\alpha$ whose preimages under all inclusions in_α, $\alpha \in A$, are open is a topological structure.

The sum $\bigsqcup_{\alpha\in A} X_\alpha$ with this topology is the (*disjoint*) *sum of the topological spaces* X_α $(\alpha \in A)$.

22.L. The topology described in *22.K* is the finest topology with respect to which all inclusions in_α are continuous.

22.17. The maps $\mathrm{in}_\beta : X_\beta \to \bigsqcup_{\alpha\in A} X_\alpha$ are topological embeddings, and their images are both open and closed in $\bigsqcup_{\alpha\in A} X_\alpha$.

22.18. Which of the standard topological properties are inherited from summands X_α by the sum $\bigsqcup_{\alpha\in A} X_\alpha$? Which are not?

⌈22′13⌋ Attaching Space

Let X and Y be two spaces, A a subset of Y, and $f : A \to X$ a continuous map. The quotient space $X \cup_f Y = (X \sqcup Y)/_{[a \sim f(a) \text{ for } a \in A]}$ is called the result of *attaching* or *gluing* the space Y to the space X via f. The map f is the *attaching map*.

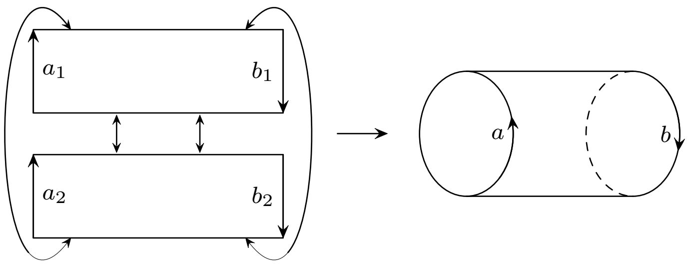
Here the partition of $X \sqcup Y$ consists of singletons in $\mathrm{in}_2(Y \smallsetminus A)$ and $\mathrm{in}_1(X \smallsetminus f(A))$, and sets $\mathrm{in}_1(x) \cup \mathrm{in}_2\big(f^{-1}(x)\big)$ with $x \in f(A)$.

22.19. Prove that the composition of the inclusion $X \to X \sqcup Y$ and the projection $X \sqcup Y \to X \cup_f Y$ is a topological embedding.

22.20. Prove that if X is a point, then $X \cup_f Y$ is $Y/_A$.

22.M. Prove that attaching the n-disk D^n to its copy via the identity map of the boundary sphere S^{n-1} we obtain a space homeomorphic to S^n.

22.21. Prove that the Klein bottle is a result of gluing together two copies of the Möbius strip via the identity map of the boundary circle.



22.22. Prove that the result of gluing together two copies of a cylinder via the identity map of the boundary circles (of one copy to the boundary circles of the other) is homeomorphic to $S^1 \times S^1$.

22.23. Prove that the result of gluing together two copies of the solid torus $S^1 \times D^2$ via the identity map of the boundary torus $S^1 \times S^1$ is homeomorphic to $S^1 \times S^2$.

22.24. Obtain the Klein bottle by gluing two copies of the cylinder $S^1 \times I$ to each other.

22.25. Prove that the result of gluing together two copies of the solid torus $S^1 \times D^2$ via the map

$$S^1 \times S^1 \to S^1 \times S^1 : (x, y) \mapsto (y, x)$$

of the boundary torus to its copy is homeomorphic to S^3.

22.N. Let X and Y be two spaces, A a subset of Y, and $f, g : A \to X$ two continuous maps. Prove that if there exists a homeomorphism $h : X \to X$ such that $h \circ f = g$, then $X \cup_f Y$ and $X \cup_g Y$ are homeomorphic.

22.O. Prove that $D^n \cup_h D^n$ is homeomorphic to S^n for each homeomorphism $h : S^{n-1} \to S^{n-1}$.

22.26. Classify up to homeomorphism the spaces that can be obtained from a square by identifying a pair of opposite sides by a homeomorphism.

22.27. Classify up to homeomorphism the spaces that can be obtained from two copies of $S^1 \times I$ by identifying the copies of $S^1 \times \{0, 1\}$ via a homeomorphism.

22.28. Prove that the topological type of the space resulting from gluing together two copies of the Möbius strip via a homeomorphism of the boundary circle does not depend on the homeomorphism.

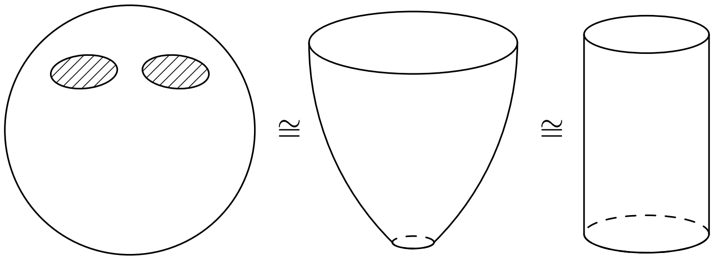
22.29. Classify up to homeomorphism the spaces that can be obtained from $S^1 \times I$ by identifying $S^1 \times 0$ and $S^1 \times 1$ via a homeomorphism.

⌈22′14⌋ Basic Surfaces

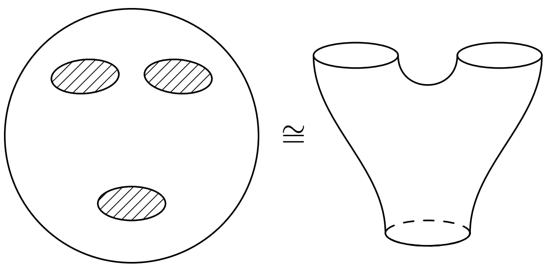
Deleting from the torus $S^1 \times S^1$ the interior of an embedded disk, we obtain a *handle*. Similarly, deleting from the two-sphere the interior of n disjoint embedded disks, we obtain a *sphere with n holes*.

22.P. A sphere with a hole is homeomorphic to the disk D^2.

22.Q. A sphere with two holes is homeomorphic to the cylinder $S^1 \times I$.



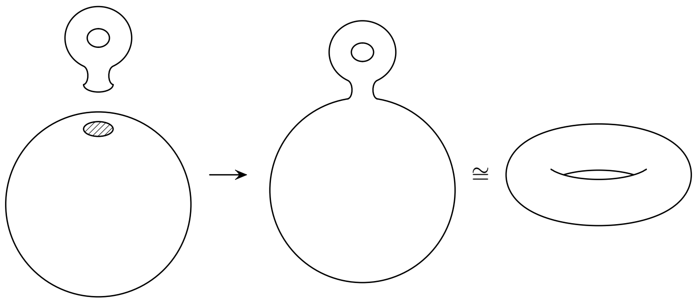
A sphere with three holes has a special name. It is called *pantaloons* or just *pants* .



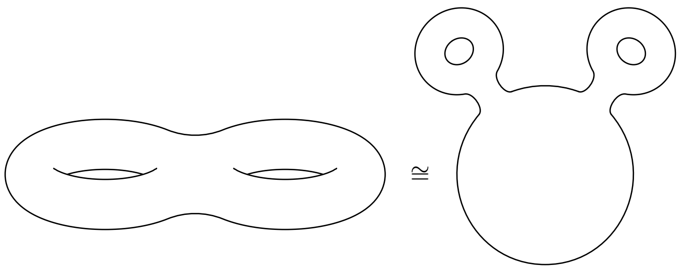
The result of attaching p copies of a handle to a sphere with p holes via embeddings homeomorphically mapping the boundary circles of the handles onto those of the holes is a *sphere with p handles*, or, in a more ceremonial way (and less understandable, for a while), an *orientable connected closed surface of genus p*.

22.30. Prove that a sphere with p handles is well defined up to homeomorphism (i.e., the topological type of the result of gluing does not depend on the attaching embeddings).

22.R. A sphere with one handle is homeomorphic to the torus $S^1 \times S^1$.



22.S. A sphere with two handles is homeomorphic to the result of gluing together two copies of a handle via the identity map of the boundary circle.



A sphere with two handles is a *pretzel*. Sometimes, this word also denotes a sphere with more handles.

The space obtained from a sphere with q holes by attaching q copies of the Möbius strip via embeddings of the boundary circles of the Möbius strips onto the boundary circles of the holes (the boundaries of the holes) is a *sphere with q cross-caps*, or a *nonorientable connected closed surface of genus q*.

22.31. Prove that a sphere with q cross-caps is well defined up to homeomorphism (i.e., the topological type of the result of gluing does not depend on the attaching embeddings).

22.T. A sphere with a cross-cap is homeomorphic to the projective plane.

22.U. A sphere with two cross-caps is homeomorphic to the Klein bottle.

A sphere, spheres with handles, and spheres with cross-caps are *basic surfaces*.

22.V. Prove that a sphere with p handles and q cross-caps is homeomorphic to a sphere with $2p + q$ cross-caps (here $q > 0$).

22.32. Classify up to homeomorphism those spaces which are obtained by attaching p copies of $S^1 \times I$ to a sphere with $2p$ holes via embeddings of the boundary circles of the cylinders onto the boundary circles of the sphere with holes.

23. Projective Spaces

This section can be considered as a continuation of the previous one. The quotient spaces described here are of too great importance to regard them just as examples of quotient spaces.

⌈23′1⌋ Real Projective Space of Dimension n

This space is defined as the quotient space of the sphere S^n by the partition into pairs of antipodal points, and denoted by $\mathbb{R}P^n$.

23.A. *The space $\mathbb{R}P^n$ is homeomorphic to the quotient space of the n-disk D^n by the partition into singletons in the interior of D^n, and pairs of antipodal point of the boundary sphere S^{n-1}.*

23.B. $\mathbb{R}P^0$ is a point.

23.C. The space $\mathbb{R}P^1$ is homeomorphic to the circle S^1.

23.D. The space $\mathbb{R}P^2$ is homeomorphic to the projective plane defined in the previous section.

23.E. *The space $\mathbb{R}P^n$ is canonically homeomorphic to the quotient space of $\mathbb{R}^{n+1} \smallsetminus 0$ by the partition into one-dimensional vector subspaces of $\mathbb{R}^{n+1}$ punctured at* 0.

A point of the space $\mathbb{R}^{n+1} \smallsetminus 0$ is a sequence of real numbers, which are not all zeros. These numbers are the ***homogeneous coordinates*** of the corresponding point of $\mathbb{R}P^n$. The point with homogeneous coordinates x_0, x_1, $\dots$, x_n is denoted by $(x_0 : x_1 : \dots : x_n)$. Homogeneous coordinates determine a point of $\mathbb{R}P^n$, but are not determined by this point: proportional vectors of coordinates $(x_0, x_1, \dots, x_n)$ and $(\lambda x_0, \lambda x_1, \dots, \lambda x_n)$ determine the same point of $\mathbb{R}P^n$.

23.F. *The space $\mathbb{R}P^n$ is canonically homeomorphic to the metric space whose points are lines of $\mathbb{R}^{n+1}$ through the origin $0 = (0, \dots, 0)$ and the metric is defined as the angle between lines (which takes values in $[0, \pi/2]$). Prove that this is really a metric.*

23.G. Prove that the map

$$i : \mathbb{R}^n \to \mathbb{R}P^n : (x_1, \dots, x_n) \mapsto (1 : x_1 : \dots : x_n)$$

is a topological embedding. What is its image? What is the inverse map of its image onto $\mathbb{R}^n$?

23.H. Construct a topological embedding $\mathbb{R}P^{n-1} \to \mathbb{R}P^n$ with image $\mathbb{R}P^n \smallsetminus i(\mathbb{R}^n)$, where i is the embedding from Problem *23.G*.

Therefore, the projective space $\mathbb{R}P^n$ can be regarded as the result of extending $\mathbb{R}^n$ by adjoining "improper" or "infinite" points, which constitute a projective space $\mathbb{R}P^{n-1}$.

23.1. Introduce a natural topological structure in the set of all lines on the plane and prove that the resulting space is homeomorphic to a) $\mathbb{R}P^2 \smallsetminus \{\text{pt}\}$; b) open Möbius strip (i.e., a Möbius strip with the boundary circle removed).

23.2. Prove that the set of all rotations of the space $\mathbb{R}^3$ around lines passing through the origin equipped with the natural topology is homeomorphic to $\mathbb{R}P^3$.

⌈23′2x⌋ Complex Projective Space of Dimension n

This space is defined as the quotient space of the unit sphere S^{2n+1} in $\mathbb{C}^{n+1}$ by the partition into circles cut by (complex) lines of $\mathbb{C}^{n+1}$ passing through the point 0. It is denoted by $\mathbb{C}P^n$.

23.Ix. $\mathbb{C}P^n$ is homeomorphic to the quotient space of the unit $2n$-disk D^{2n} of the space $\mathbb{C}^n$ by the partition whose elements are singletons in the interior of D^{2n} and circles cut on the boundary sphere S^{2n-1} by (complex) lines of $\mathbb{C}^n$ passing through the origin $0 \in \mathbb{C}^n$.

23.Jx. $\mathbb{C}P^0$ is a point.

The space $\mathbb{C}P^1$ is a *complex projective line*.

23.Kx. The complex projective line $\mathbb{C}P^1$ is homeomorphic to S^2.

23.Lx. The space $\mathbb{C}P^n$ is canonically homeomorphic to the quotient space of the space $\mathbb{C}^{n+1} \smallsetminus 0$ by the partition into complex lines of $\mathbb{C}^{n+1}$ punctured at 0.

Hence, $\mathbb{C}P^n$ can be regarded as the space of complex-proportional nonzero complex sequences $(x_0, x_1, \dots, x_n)$. The notation $(x_0 : x_1 : \dots : x_n)$ and the term *homogeneous coordinates* introduced in the real case are used in the same way for the complex case.

23.Mx. The space $\mathbb{C}P^n$ is canonically homeomorphic to the metric space, whose points are the (complex) lines of $\mathbb{C}^{n+1}$ passing through the origin 0, and the metric is defined as the angle between lines (which takes values in $[0, \pi/2]$).

⌈23′3x⌋ Quaternionic Projective Spaces

Recall that $\mathbb{R}^4$ bears a remarkable multiplication, which was discovered by R. W. Hamilton in 1843. It can be defined by the formula

$$\begin{aligned}(x_1, x_1, x_3, x_4) \times (y_1, y_2, y_3, y_4) = \\ (x_1y_1 - x_2y_2 - x_3y_3 - x_4y_4, \quad x_1y_2 + x_2y_1 + x_3y_4 - x_4y_3, \\ x_1y_3 - x_2y_4 + x_3y_1 + x_4y_2, \quad x_1y_4 + x_2y_3 - x_3y_2 + x_4y_1).\end{aligned}$$

It is bilinear, and to describe it in a shorter way it suffices to specify the products of the basis vectors. Following Hamilton, the latter are traditionally denoted (in this case) as follows:

$$1 = (1,0,0,0), \quad i = (0,1,0,0), \quad j = (0,0,1,0), \quad \text{and} \quad k = (0,0,0,1).$$

In this notation, 1 is really a unity: $(1,0,0,0) \times x = x$ for each $x \in \mathbb{R}^4$. The rest of the multiplication table looks as follows:

$$ij = k, \quad jk = i, \quad ki = j, \quad ji = -k, \quad kj = -i, \quad \text{and} \quad ik = -j.$$

Together with coordinate-wise addition, this multiplication determines a structure of algebra in $\mathbb{R}^4$. Its elements are *quaternions*.

23.Nx. Check that the quaternion multiplication is associative.

It is not commutative (e.g., $ij = k \neq -k = ji$). Otherwise, quaternions are very similar to complex numbers. As in $\mathbb{C}$, there is a transformation called *conjugation* acting in the set of quaternions. As well as the conjugation of complex numbers, it is also denoted by a bar: $x \mapsto \overline{x}$. It is defined by the formula $(x_1, x_2, x_3, x_4) \mapsto (x_1, -x_2, -x_3, -x_4)$ and has two remarkable properties:

23.Ox. We have $\overline{ab} = \overline{b}\,\overline{a}$ for any two quaternions a and b.

23.Px. We have $a\overline{a} = |a|^2$, i.e., the product of any quaternion a by the conjugate quaternion $\overline{a}$ equals $(|a|^2, 0, 0, 0)$.

The latter property allows us to define, for any $a \in \mathbb{R}^4$, the inverse quaternion

$$a^{-1} = |a|^{-2}\overline{a}$$

such that $aa^{-1} = 1$.

Hence, the quaternion algebra is a *division algebra* or a *skew field*. It is denoted by $\mathbb{H}$ after Hamilton, who discovered it.

In the space $\mathbb{H}^n = \mathbb{R}^{4n}$, there are right quaternionic lines, i.e., subsets $\{(a_1\xi, \dots, a_n\xi) \mid \xi \in \mathbb{H}\}$, and similar left quaternionic lines $\{(\xi a_1, \dots, \xi a_n) \mid \xi \in \mathbb{H}\}$. Each of them is a real 4-dimensional subspace of $\mathbb{H}^n = \mathbb{R}^{4n}$.

23.Qx. Find a right quaternionic line that is not a left quaternionic line.

23.Rx. Prove that two right quaternionic lines in $\mathbb{H}^n$ either meet only at 0, or coincide.

The quotient space of the unit sphere S^{4n+3} of the space $\mathbb{H}^{n+1} = \mathbb{R}^{4n+4}$ by the partition into its intersections with right quaternionic lines is the (*right*) *quaternionic projective space of dimension* n. Similarly, but with left quaternionic lines, we define the (*left*) *quaternionic projective space of dimension* n.

***23.S*x.** Are the right and left quaternionic projective space of the same dimension homeomorphic?

The left quaternionic projective space of dimension n is denoted by $\mathbb{H}P^n$.

***23.T*x.** $\mathbb{H}P^0$ is a singleton.

***23.U*x.** $\mathbb{H}P^n$ is homeomorphic to the quotient space of the closed unit disk D^{4n} in $\mathbb{H}^n$ by the partition into points of the interior of D^{4n} and the 3-spheres that are intersections of the boundary sphere S^{4n-1} with (left quaternionic) lines of $\mathbb{H}^n$.

The space $\mathbb{H}P^1$ is the *quaternionic projective line*.

***23.V*x.** Quaternionic projective line $\mathbb{H}P^1$ is homeomorphic to S^4.

***23.W*x.** $\mathbb{H}P^n$ is canonically homeomorphic to the quotient space of $\mathbb{H}^{n+1} \smallsetminus 0$ by the partition to left quaternionic lines of $\mathbb{H}^{n+1}$ passing through the origin and punctured at it.

Hence, $\mathbb{H}P^n$ can be presented as the space of classes of left proportional (in the quaternionic sense) nonzero sequences $(x_0, \dots, x_n)$ of quaternions. The notation $(x_0 : x_1 : \dots : x_n)$ and the term *homogeneous coordinates* introduced above in the real case are used in the same way in the quaternionic situation.

***23.X*x.** $\mathbb{H}P^n$ is canonically homeomorphic to the set of (left quaternionic) lines of $\mathbb{H}^{n+1}$ equipped with the topology generated by the angular metric (which takes values in $\left[0, \pi/2\right]$).

24x. Finite Topological Spaces

⌈24′1x⌋ Set-Theoretic Digression: Splitting a Transitive Relation Into Equivalence and Partial Order

In the definitions of equivalence and partial order relations, the condition of transitivity seems to be the most important. Below, we supply a formal justification of this feeling by showing that the other conditions are natural companions of transitivity, although they are not its consequences.

24.Ax. *Let $\prec$ be a transitive relation on a set X. Then the relation $\precsim$ defined by*

$$a \precsim b \text{ if } a \prec b \text{ or } a = b$$

is also transitive (and, furthermore, it is certainly reflexive, i.e., $a \precsim a$ for each $a \in X$).

A binary relation $\precsim$ on a set X is a ***preorder*** if it is transitive and reflective, i.e., satisfies the following conditions:

- *Transitivity*. If $a \precsim b$ and $b \precsim c$, then $a \precsim c$.
- *Reflexivity*. We have $a \precsim a$ for any a.

A set X equipped with a preorder is ***preordered***.

If a preorder is antisymmetric, then this is a nonstrict order.

24.1x. Is the relation $a \mid b$ a preorder on the set $\mathbb{Z}$ of integers?

24.Bx. *If $(X, \precsim)$ is a preordered set, then the relation $\sim$ defined by*

$$a \sim b \text{ if } a \precsim b \text{ and } b \precsim a$$

is an equivalence relation (i.e., it is symmetric, reflexive, and transitive) on X.

24.2x. What equivalence relation is defined on $\mathbb{Z}$ by the preorder $a \mid b$?

24.Cx. *Let $(X, \precsim)$ be a preordered set, and let $\sim$ be an equivalence relation defined on X by $\precsim$ according to 24.Bx. Then $a' \sim a$, $a \precsim b$, and $b \sim b'$ imply $a' \precsim b'$ and in this way $\precsim$ determines a relation on the set of equivalence classes $X/_\sim$. This relation is a nonstrict partial order.*

Thus, any transitive relation generates an equivalence relation and a partial order on the set of equivalence classes.

24.Dx. How this chain of constructions would degenerate if the original relation was

(1) an equivalence relation, or

(2) nonstrict partial order?

24.Ex. *In any topological space, the relation* $\precsim$ *defined by*

$$a \precsim b \text{ if } a \in \operatorname{Cl}\{b\}$$

is a preorder.

24.3x. In the set of all subsets of an arbitrary topological space, the relation

$$A \precsim B \text{ if } A \subset \operatorname{Cl} B$$

is a preorder. This preorder determines the following equivalence relation: two sets are equivalent iff they have the same closure.

24.Fx. *The equivalence relation determined by the preorder which is defined in Theorem 24.Ex determines the partition of the space into maximal (with respect to inclusion) indiscrete subspaces. The quotient space satisfies the Kolmogorov separation axiom* T_0*.*

The quotient space of Theorem *24.Fx* is the *maximal* T_0*-quotient of* X.

24.Gx. A continuous image of an indiscrete space is indiscrete.

24.Hx. Prove that any continuous map $X \to Y$ induces a continuous map of the maximal T_0-quotient of X to the maximal T_0-quotient of Y.

⌈24′2x⌋ The Structure of Finite Topological Spaces

The results of the preceding subsection provide a key to understanding the structure of finite topological spaces. Let X be a finite space. By Theorem *24.Fx*, X is partitioned to indiscrete clusters of points. By *24.Gx*, continuous maps between finite spaces respect these clusters and, by *24.Hx*, induce continuous maps between the maximal T_0-quotient spaces.

This means that we can consider a finite topological space as its maximal T_0-quotient whose points are equipped with multiplicities, which are positive integers: the numbers of points in the corresponding clusters of the original space.

The maximal T_0-quotient of a finite space is a smallest neighborhood space (as a finite space). By Theorem *15.O*, its topology is determined by a partial order. By Theorem *10.Xx*, homeomorphisms between spaces with poset topologies are monotone bijections.

Thus, a finite topological space is characterized up to homeomorphism by a finite poset whose elements are equipped with multiplicities (positive integers). Two such spaces are homeomorphic iff there exists a monotone bijection between the corresponding posets that preserves the multiplicities. To recover the topological space from a poset with multiplicities, we must equip the poset with the poset topology and then replace each of its elements by an indiscrete cluster of points, the number points in which is the multiplicity of the element.

⌈24′3x⌋ Simplicial Schemes

Let V be a set, Σ a certain set of subsets of V. A pair (V, Σ) is a *simplicial scheme* with the set of *vertices* V and the set of *simplices* Σ if

- each subset of each set in Σ belongs to Σ,
- the intersection of any collection of sets in Σ belongs to Σ,
- each singleton in V belongs to Σ.

The set Σ is partially ordered by inclusion. When equipped with the poset topology of this partial order, it is called *the space of simplices* of the simplicial scheme (X, Σ).

A simplicial scheme also yields another topological space. Namely, for a simplicial scheme (V, Σ), consider the set $S(V, \Sigma)$ of all functions $c : V \to [0, 1]$ such that

$$\operatorname{Supp}(c) = \{v \in V \mid c(v) \neq 0\} \in \Sigma$$

and $\sum_{v \in V} c(v) = 1$. Equip $S(V, \Sigma)$ with the topology generated by metric

$$\rho(c_1, c_2) = \sup_{v \in V} |c_1(v) - c_2(v)|.$$

The space $S(V, \Sigma)$ is a *simplicial* or *triangulated* space. It is covered by the sets $\{c \in S \mid \operatorname{Supp}(c) = \sigma\}$, where $\sigma \in \Sigma$, which are called its (*open*) *simplices*.

24.4x. Which open simplices of a simplicial space are open sets, which are closed, and which are neither closed nor open?

24.Ix. For each $\sigma \in \Sigma$, find a homeomorphism of the space

$$\{c \in S \mid \operatorname{Supp}(c) = \sigma\} \subset S(V, \Sigma)$$

onto an open simplex whose dimension is one less than the number of vertices belonging to σ. (Recall that the open n-simplex is the set $\{(x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1} \mid x_j > 0 \text{ for } j = 1, \dots, n+1 \text{ and } \sum_{i=1}^{n+1} x_i = 1\}$.)

24.Jx. Prove that for each simplicial scheme (V, Σ) the quotient space of the simplicial space $S(V, \Sigma)$ by its partition into open simplices is homeomorphic to the space Σ of simplices of the simplicial scheme (V, Σ).

⌈24′4x⌋ Barycentric Subdivision of a Poset

24.Kx. Find a poset which is not isomorphic to the set of simplices (ordered by inclusion) of whatever simplicial scheme.

Let $(X, \prec)$ be a poset. Consider the set X' of all nonempty finite strictly increasing sequences $a_1 \prec a_2 \prec \dots \prec a_n$ of elements of X. It can also be described as the set of all nonempty finite subsets of X in each of which $\prec$ determines a linear order. It is naturally ordered by inclusion.

The poset $(X', \subset)$ is the *barycentric subdivision of* $(X, \prec)$.

24.Lx. For any poset $(X, \prec)$, the pair (X, X') is a simplicial scheme.

There is a natural map $X' \to X$ that sends an element of X' (i.e., a nonempty finite linearly ordered subset of X) to its greatest element.

24.Mx. Is this map monotone? Strictly monotone? The same questions concerning a similar map that sends a nonempty finite linearly ordered subset of X to its smallest element.

Let (V, Σ) be a simplicial scheme, and let Σ' be the barycentric subdivision of Σ (ordered by inclusion). The simplicial scheme (Σ, Σ') is the *barycentric subdivision* of the simplicial scheme (V, Σ).

There is a natural mapping $\Sigma \to S(V, \Sigma)$ that sends a simplex $\sigma \in \Sigma$ (i.e., a subset $\{v_0, v_1, \dots, v_n\}$ of V) to the function $b_\sigma : V \to \mathbb{R}$ with $b_\sigma(v_i) = 1/(n+1)$ and $b_\sigma(v) = 0$ for any $v \notin \sigma$.

Define a map $\beta : S(\Sigma, \Sigma') \to S(V, \Sigma)$ that sends a function $\varphi : \Sigma \to \mathbb{R}$ to the function

$$V \to \mathbb{R} : v \mapsto \sum_{\sigma \in \Sigma} \varphi(\sigma) b_\sigma(v).$$

24.Nx. Prove that the map $\beta : S(\Sigma, \Sigma') \to S(V, \Sigma)$ is a homeomorphism and constitutes, together with the projections $S(V, \Sigma) \to \Sigma$ and $S(\Sigma, \Sigma') \to \Sigma'$ and the natural map $\Sigma' \to \Sigma$, a commutative diagram

$$\begin{array}{ccc} S(\Sigma, \Sigma') & \xrightarrow{\beta} & S(V, \Sigma) \\ \downarrow & & \downarrow \\ \Sigma' & \longrightarrow & \Sigma \end{array}$$

25x. Spaces of Continuous Maps

⌈25′1x⌋ Sets of Continuous Mappings

We denote by $\mathcal{C}(X,Y)$ the set of all continuous maps of a space X to a space Y.

***25.1*x.** Let X be nonempty. Prove that $\mathcal{C}(X,Y)$ is a singleton iff so is Y.

***25.2*x.** Let X be nonempty. Prove that there exists an injection $Y \to \mathcal{C}(X,Y)$. In other words, the cardinality $\operatorname{card}\mathcal{C}(X,Y)$ of $\mathcal{C}(X,Y)$ is greater than or equal to $\operatorname{card}Y$.

25.3*x. *Riddle. Find natural conditions implying that $\mathcal{C}(X,Y) = Y$.

***25.4*x.** Let $Y = \{0,1\}$ be equipped with the topology $\{\varnothing, \{0\}, \{0,1\}\}$. Prove that there exists a bijection between $\mathcal{C}(X,Y)$ and the topological structure of X.

***25.5*x.** Let X be an n-element discrete space. Prove that $\mathcal{C}(X,Y)$ can be identified with $Y \times \dots \times Y$ (n factors).

***25.6*x.** Let Y be a k-element discrete space. Find a necessary and sufficient condition for the set $\mathcal{C}(X,Y)$ to contain k^2 elements.

⌈25′2x⌋ Topologies on a Set of Continuous Mappings

Let X and Y be two topological spaces, $A \subset X$, and $B \subset Y$. We define $W(A,B) = \{f \in \mathcal{C}(X,Y) \mid f(A) \subset B\}$,

$$\Delta^{(pw)} = \{W(a,U) \mid a \in X,\ U \text{ is open in } Y\},$$

and

$$\Delta^{(co)} = \{W(C,U) \mid C \subset X \text{ is compact, } U \text{ is open in } Y\}.$$

***25.A*x.** $\Delta^{(pw)}$ is a subbase of a topological structure on $\mathcal{C}(X,Y)$.

The topological structure generated by $\Delta^{(pw)}$ is the *topology of pointwise convergence*. The set $\mathcal{C}(X,Y)$ equipped with this structure is denoted by $\mathcal{C}^{(pw)}(X,Y)$.

***25.B*x.** $\Delta^{(co)}$ is a subbase of a topological structures on $\mathcal{C}(X,Y)$.

The topological structure determined by $\Delta^{(co)}$ is the *compact-open topology*. Hereafter we denote by $\mathcal{C}(X,Y)$ the space of all continuous maps $X \to Y$ with the compact-open topology, unless the contrary is specified explicitly.

25.C*x *Compact-Open Versus Pointwise. The compact-open topology is finer than the topology of pointwise convergence.

***25.7*x.** Prove that $\mathcal{C}(I,I)$ is not homeomorphic to $\mathcal{C}^{(pw)}(I,I)$.

Denote by $Const(X,Y)$ the set of all constant maps $f : X \to Y$.

***25.8*x.** Prove that the topology of pointwise convergence and the compact-open topology of $\mathcal{C}(X,Y)$ induce the same topological structure on $Const(X,Y)$, which, with this topology, is homeomorphic Y.

***25.9*x.** Let X be an n-element discrete space. Prove that $\mathcal{C}^{(pw)}(X,Y)$ is homeomorphic $Y \times \dots \times Y$ (n times). Is this true for $\mathcal{C}(X,Y)$?

⌈25′3x⌋ Topological Properties of Mapping Spaces

***25.D*x.** If Y is Hausdorff, then $\mathcal{C}^{(pw)}(X,Y)$ is Hausdorff for any space X. Is this true for $\mathcal{C}(X,Y)$?

***25.10*x.** Prove that $\mathcal{C}(I,X)$ is path-connected iff so is X.

***25.11*x.** Prove that $\mathcal{C}^{(pw)}(I,I)$ is not compact. Is the space $\mathcal{C}(I,I)$ compact?

⌈25′4x⌋ Metric Case

***25.E*x.** If Y is metrizable and X is compact, then $\mathcal{C}(X,Y)$ is metrizable.

Let (Y,ρ) be a metric space, X a compact space. For continuous maps $f,g : X \to Y$, let

$$d(f,g) = \max\{\rho(f(x),g(x)) \mid x \in X\}.$$

25.F*x *This is a Metric. If X is a compact space and Y a metric space, then d is a metric on the set $\mathcal{C}(X,Y)$.

Let X be a topological space, Y a metric space with metric ρ. A sequence f_n of maps $X \to Y$ *uniformly converges* to $f : X \to Y$ if for each $\varepsilon > 0$ there exists a positive integer N such that $\rho(f_n(x), f(x)) < \varepsilon$ for any $n > N$ and $x \in X$. This is a straightforward generalization of the notion of uniform convergence which is known from Calculus.

25.G*x *Metric of Uniform Convergence. Let X be a compact space, (Y,d) a metric space. A sequence f_n of maps $X \to Y$ converges to $f : X \to Y$ in the topology generated by d iff f_n uniformly converges to f.

***25.H*x *Completeness of* $\mathcal{C}(X,Y)$.** Let X be a compact space, (Y,ρ) a complete metric space. Then $\big(\mathcal{C}(X,Y), d\big)$ is a complete metric space.

25.I*x *Uniform Convergence Versus Compact-Open. Let X be a compact space, Y a metric space. Then the topology generated by d on $\mathcal{C}(X,Y)$ is the compact-open topology.

***25.12*x.** Prove that the space $\mathcal{C}(\mathbb{R},I)$ is metrizable.

***25.13*x.** Let Y be a bounded metric space, and let X be a topological space admitting a presentation $X = \bigcup_{i=1}^{\infty} X_i$, where X_i is compact and $X_i \subset \operatorname{Int} X_{i+1}$ for each $i = 1, 2, \dots$. Prove that $\mathcal{C}(X,Y)$ is metrizable.

Denote by $\mathcal{C}_b(X,Y)$ the set of all continuous bounded maps from a topological space X to a metric space Y. For maps $f,g \in \mathcal{C}_b(X,Y)$, put

$$d^\infty(f,g) = \sup\{\rho(f(x),g(x)) \mid x \in X\}.$$

25.Jx Metric on Bounded Maps. This is a metric on $\mathcal{C}_b(X,Y)$.

25.Kx d^∞ and Uniform Convergence. Let X be a topological space, Y a metric space. A sequence f_n of bounded maps $X \to Y$ converges to $f : X \to Y$ in the topology generated by d^∞ iff f_n uniformly converge to f.

25.Lx When Uniform Is not Compact-Open. Find X and Y such that the topology generated by d^∞ on $\mathcal{C}_b(X,Y)$ is not the compact-open topology.

⌈25′5x⌋ Interactions with Other Constructions

25.Mx. For any continuous maps $\varphi : X' \to X$ and $\psi : Y \to Y'$, the map $\mathcal{C}(X,Y) \to \mathcal{C}(X',Y') : f \mapsto \psi \circ f \circ \varphi$ is continuous.

25.Nx Continuity of Restricting. Let X and Y be two spaces, $A \subset X$ a subset. Prove that the map $\mathcal{C}(X,Y) \to \mathcal{C}(A,Y) : f \mapsto f|_A$ is continuous.

25.Ox Extending Target. For any spaces X and Y and any $B \subset Y$, the map $\mathcal{C}(X,B) \to \mathcal{C}(X,Y) : f \mapsto i_B \circ f$ is a topological embedding.

25.Px Maps to Product. For any three spaces X, Y, and Z, the space $\mathcal{C}(X, Y \times Z)$ is canonically homeomorphic to $\mathcal{C}(X,Y) \times \mathcal{C}(X,Z)$.

25.Qx Restricting to Sets Covering Source. Let $\{X_1,\dots,X_n\}$ be a closed cover of X. Prove that for each space Y the map

$$\phi : \mathcal{C}(X,Y) \to \prod_{i=1}^{n} \mathcal{C}(X_i,Y) \ : \ f \mapsto (f|_{X_1},\dots,f|_{X_n})$$

is a topological embedding. What if the cover is not fundamental?

25.Rx. Riddle. Can you generalize assertion *25.Qx*?

25.Sx Continuity of Composing. Let X be a space, Y a locally compact Hausdorff space. Prove that the map

$$\mathcal{C}(X,Y) \times \mathcal{C}(Y,Z) \to \mathcal{C}(X,Z) \ : \ (f,g) \mapsto g \circ f$$

is continuous.

25.14x. Is local compactness of Y necessary in *25.Sx*?

25.Tx Factorizing Source. Let S be a closed partition[2] of a Hausdorff compact space X. Prove that for any space Y the map

$$\phi : \mathcal{C}(X/S,Y) \to \mathcal{C}(X,Y)$$

is a topological embedding.

[2]Recall that a partition is *closed* if the saturation of each closed set is closed.

***25.15*x.** Are the conditions imposed on S and X in *25.Tx* necessary?

25.Ux The Evaluation Map. Let X and Y be two spaces. Prove that if X is locally compact and Hausdorff, then the map

$$\phi : \mathcal{C}(X, Y) \times X \to Y \ : \ (f, x) \mapsto f(x)$$

is continuous.

***25.16*x.** Are the conditions imposed on X in *25.Ux* necessary?

$\lceil 25'6\mathrm{x} \rfloor$ Mappings $X \times Y \to Z$ and $X \to \mathcal{C}(Y, Z)$

***25.V*x.** Let X, Y, and Z be three topological spaces, $f : X \times Y \to Z$ a continuous map. Then the map

$$F : X \to \mathcal{C}(Y, Z) \ : \ F(x) : y \mapsto f(x, y),$$

is continuous.

The converse assertion is also true under certain additional assumptions.

***25.W*x.** Let X and Z be two spaces, Y a Hausdorff locally compact space, $F : X \to \mathcal{C}(Y, Z)$ a continuous map. Then the map $f : X \times Y \to Z : (x, y) \mapsto F(x)(y)$ is continuous.

***25.X*x.** If X is a Hausdorff space and the collection $\Sigma_Y = \{U_\alpha\}$ is a subbase of the topological structure of Y, then the collection $\{W(K, U) \mid U \in \Sigma\}$ is a subbase of the compact-open topology on $\mathcal{C}(X, Y)$.

***25.Y*x.** Let X, Y, and Z be three spaces. Let

$$\Phi : \mathcal{C}(X \times Y, Z) \to \mathcal{C}(X, \mathcal{C}(Y, Z))$$

be defined by the relation

$$\Phi(f)(x) : y \mapsto f(x, y).$$

Then

(1) if X is a Hausdorff space, then Φ is continuous;
(2) if X is a Hausdorff space, while Y is locally compact and Hausdorff, then Φ is a homeomorphism.

***25.Z*x.** *Let S be a partition of a space X, and let $\mathrm{pr} : X \to X/_S$ be the projection. The space $X \times Y$ bears a natural partition $S' = \{A \times y \mid A \in S,\ y \in Y\}$. If the space Y is Hausdorff and locally compact, then the natural quotient map $f : (X \times Y)/_{S'} \to X/_S \times Y$ of the projection $\mathrm{pr} \times \mathrm{id}_Y$ is a homeomorphism.*

***25.17*x.** Try to prove Theorem *25.Zx* directly.

Chapter V

Topological Algebra

In this chapter, we study topological spaces strongly related to groups: either the space itself is a group in a nice way (so that all the maps coming from group theory are continuous), or a group acts on a topological space and can be thought of as consisting of homeomorphisms.

This material has interdisciplinary character. Although it plays important roles in many areas of Mathematics, it is not so important in the framework of general topology. Quite often, this material can be postponed till the introductory chapters of the mathematical courses that really require it (functional analysis, Lie groups, etc.). In the framework of general topology, this material provides a great collection of exercises.

In the second part of the book, which is devoted to algebraic topology, groups appear in a more profound way. So, the reader will meet groups no later than the next chapter, when studying fundamental groups.

Groups are attributed to algebra. In the mathematics built on sets, main objects are sets with additional structure. Above, we met a few of the most fundamental of these structures: topology, metric, and (partial) order. Topology and metric evolved from geometric considerations. Algebra studied algebraic operations with numbers and similar objects and introduced into the set-theoretic Mathematics various structures based on operations. One of the simplest (and most versatile) of these structures is the structure of a group. It emerges in an overwhelming majority of mathematical environments. It often appears together with topology and in a nice interaction with it. This interaction is a subject of topological algebra.

The second part of this book is called Algebraic Topology. It also treats the interaction of topology and algebra, spaces and groups. But this is a completely different interaction. There the structures of topological space and group do not live on the same set, but the group encodes topological properties of the space.

26x. Generalities on Groups

This section is included mainly to recall the most elementary definitions and statements concerning groups. We do not mean to present a self-contained outline of the group theory. The reader is actually assumed to be familiar with groups, homomorphisms, subgroups, quotient groups, etc.

If this is not yet so, we recommend reading one of the numerous algebraic textbooks covering the elementary group theory. The mathematical culture, which must be acquired for mastering the material presented previously in this book, would make this an easy and pleasant exercise.

As a temporary solution, the reader can read a few definitions and prove a few theorems gathered in this section. They provide a sufficient basis for most of what follows.

⌈26′1x⌋ The Notion of Group

Recall that a *group* is a set G equipped with a group operation. A *group operation* on a set G is a map $\omega : G \times G \to G$ satisfying the following three conditions (known as *group axioms*):

- **Associativity.** $\omega(a, \omega(b, c)) = \omega(\omega(a, b), c)$ for any $a, b, c \in G$.
- **Existence of Neutral Element.** There exists $e \in G$ such that $\omega(e, a) = \omega(a, e) = a$ for every $a \in G$.
- **Existence of Inverse Element.** For any $a \in G$, there exists $b \in G$ such that $\omega(a, b) = \omega(b, a) = e$.

***26.Ax* Uniqueness of Neutral Element.** *A group contains a unique neutral element.*

***26.Bx* Uniqueness of Inverse Element.** *Each element of a group has a unique inverse element.*

***26.Cx* First Examples of Groups.** In each of the following situations, check if we have a group. What is its neutral element? How to calculate the element inverse to a given one?

- The set G is the set $\mathbb{Z}$ of integers, and the group operation is addition: $\omega(a, b) = a + b$.
- The set G is the set $\mathbb{Q}_{>0}$ of positive rational numbers, and the group operation is multiplication: $\omega(a, b) = ab$.
- $G = \mathbb{R}$, and $\omega(a, b) = a + b$.
- $G = \mathbb{C}$, and $\omega(a, b) = a + b$.
- $G = \mathbb{R} \smallsetminus 0$, and $\omega(a, b) = ab$.

- G is the set of all bijections of a set A onto itself, and the group operation is composition: $\omega(a, b) = a \circ b$.

26.1x Simplest Group. 1) Can a group be empty? 2) Can it consist of one element?

A group consisting of one element is *trivial*.

26.2x Solving Equations. Let G be a set with an associative operation $\omega : G \times G \to G$. Prove that G is a group iff for any $a, b \in G$ the set G contains a unique element x such that $\omega(a, x) = b$ and a unique element y such that $\omega(y, a) = b$.

⌈26′2x⌋ Additive Versus Multiplicative

The above notation is never used! (The only exception may happen, as here, when the definition of group is discussed.) Instead, one uses either *multiplicative* or *additive* notation.

Under the multiplicative notation, the group operation is called *multiplication* and also denoted as multiplication: $(a, b) \mapsto ab$. The neutral element is called *unity* and denoted by 1 or 1_G (or e). The element inverse to a is denoted by a^{-1}. This notation is borrowed, say, from the case of nonzero rational numbers with the usual multiplication.

Under the additive notation, the group operation is called *addition* and also denoted as addition: $(a, b) \mapsto a + b$. The neutral element is called *zero* and denoted by 0. The element inverse to a is denoted by $-a$. This notation is borrowed, say, from the case of integers with the usual addition.

An operation $\omega : G \times G \to G$ is *commutative* if $\omega(a, b) = \omega(b, a)$ for any $a, b \in G$. A group with commutative group operation is *commutative* or *Abelian*. Traditionally, the additive notation is used only in the case of commutative groups, while the multiplicative notation is used both in the commutative and noncommutative cases. Below, we mostly use the multiplicative notation.

26.3x. In each of the following situations, check if we have a group:

(1) a singleton $\{a\}$ with multiplication $aa = a$,
(2) the set $\mathbb{S}_n$ of bijections of the set $\{1, 2, \dots, n\}$ of the first n positive integers onto itself with multiplication determined by composition (the *symmetric group of degree* n),
(3) the sets $\mathbb{R}^n$, $\mathbb{C}^n$, and $\mathbb{H}^n$ with coordinate-wise addition,
(4) the set $Homeo(X)$ of all homeomorphisms of a topological space X with multiplication determined by composition,
(5) the set $GL(n, \mathbb{R})$ of invertible real $n \times n$ matrices equipped with matrix multiplication,
(6) the set $M_n(\mathbb{R})$ of all real $n \times n$ matrices with addition determined by addition of matrices,

(7) the set of all subsets of a set X with multiplication determined by the symmetric difference:

$$(A, B) \mapsto A \triangle B = (A \cup B) \smallsetminus (A \cap B),$$

(8) the set $\mathbb{Z}_n$ of classes of positive integers congruent modulo n with addition determined by addition of positive integers,
(9) the set of complex roots of unity of degree n equipped with usual multiplication of complex numbers,
(10) the set $\mathbb{R}_{>0}$ of positive reals with usual multiplication,
(11) $S^1 \subset \mathbb{C}$ with standard multiplication of complex numbers,
(12) the set of translations of a plane with multiplication determined by composition.

Associativity implies that every finite sequence of elements in a group has a well-defined product, which can be calculated by a sequence of pairwise multiplications determined by any placement of parentheses, say, $abcde = (ab)(c(de))$. The distribution of the parentheses is immaterial. In the case of a three-element sequence, this is precisely the associativity: $(ab)c = a(bc)$.

***26.D*x.** Derive from the associativity that the product of any length does not depend on the position of the parentheses.

For an element a of a group G, the powers a^n with $n \in \mathbb{Z}$ are defined by the following formulas: $a^0 = 1$, $a^{n+1} = a^n a$, and $a^{-n} = (a^{-1})^n$.

***26.E*x.** Prove that raising to a power has the following properties: $a^p a^q = a^{p+q}$ and $(a^p)^q = a^{pq}$.

⌈26′3x⌋ Homomorphisms

Recall that a map $f : G \to H$ of a group to another one is a *homomorphism* if $f(xy) = f(x)f(y)$ for any $x, y \in G$.

***26.4*x.** In the above definition of a homomorphism, the multiplicative notation is used. How does this definition look in the additive notation? What if one of the groups is multiplicative, while the other is additive?

***26.5*x.** Let a be an element of a multiplicative group G. Is the map $\mathbb{Z} \to G : n \mapsto a^n$ a homomorphism?

***26.F*x.** Let G and H be two groups. Is the constant map $G \to H$ mapping the entire G to the neutral element of H a homomorphism? Is any other constant map $G \to H$ a homomorphism?

***26.G*x.** *A homomorphism maps the neutral element to the neutral element, and it maps mutually inverse elements to mutually inverse elements.*

***26.H*x.** The identity map of a group is a homomorphism. The composition of homomorphisms is a homomorphism.

Recall that a homomorphism f is an ***epimorphism*** if f is surjective, f is a ***monomorphism*** if f is injective, and f is an ***isomorphism*** if f is bijective.

26.Ix. *The map inverse to an isomorphism is also an isomorphism.*

Two groups are ***isomorphic*** if there exists an isomorphism of one of them onto another one.

26.Jx. *Isomorphism is an equivalence relation.*

> ***26.6x.*** Show that the additive group $\mathbb{R}$ is isomorphic to the multiplicative group $\mathbb{R}_{>0}$.

⌈26′4x⌋ Subgroups

A subset A of a group G is a ***subgroup*** of G if A is invariant under the group operation of G (i.e., for any $a, b \in A$ we have $ab \in A$) and A equipped with the group operation induced by that on G is a group.

For two subsets A and B of a multiplicative group G, we put $AB = \{ab \mid a \in A,\ b \in B\}$ and $A^{-1} = \{a^{-1} \mid a \in A\}$.

26.Kx. A subset A of a multiplicative group G is a subgroup of G iff $AA \subset A$ and $A^{-1} \subset A$.

> ***26.7x.*** The singleton consisting of the neutral element is a subgroup.
>
> ***26.8x.*** Prove that a subset A of a *finite* group is a subgroup if $AA \subset A$. (The condition $A^{-1} \subset A$ is superfluous in this case.)
>
> ***26.9x.*** List all subgroups of the additive group $\mathbb{Z}$.
>
> ***26.10x.*** Is $GL(n, \mathbb{R})$ a subgroup of $M_n(\mathbb{R})$? (See *26.3x* for notation.)

26.Lx. *The image of a group homomorphism $f : G \to H$ is a subgroup of H.*

26.Mx. *Let $f : G \to H$ be a group homomorphism, K a subgroup of H. Then $f^{-1}(K)$ is a subgroup of G.*

In short: The preimage of a subgroup under a group homomorphism is a subgroup.

The preimage of the neutral element under a group homomorphism $f : G \to H$ is called the ***kernel*** of f and denoted by $\operatorname{Ker} f$.

26.Nx Corollary of 26.Mx. *The kernel of a group homomorphism is a subgroup.*

26.Ox. *A group homomorphism is a monomorphism iff its kernel is trivial.*

26.Px. *The intersection of any collection of subgroups of a group is also a subgroup.*

A subgroup H of a group G is ***generated*** by a subset $S \subset G$ if H is the smallest subgroup of G containing S.

26.Qx. The subgroup H generated by S is the intersection of all subgroups of G that contain S. On the other hand, H is the set of all elements that are products of elements in S and elements inverse to elements in S.

The elements of a set that generates G are ***generators*** of G. A group generated by one element is ***cyclic***.

26.Rx. A cyclic (multiplicative) group consists of powers of its generator (i.e., if G is a cyclic group and a generates G, then $G = \{a^n \mid n \in \mathbb{Z}\}$). Any cyclic group is commutative.

26.11x. A group G is cyclic iff there exists an epimorphism $f : \mathbb{Z} \to G$.

26.Sx. *A subgroup of a cyclic group is cyclic.*

The number of elements in a group G is the ***order*** of G. It is denoted by $|G|$.

26.Tx. Let G be a finite cyclic group, d a positive divisor of $|G|$. Then G contains a unique subgroup H with $|H| = d$.

Each element of a group generates a cyclic subgroup, which consists of all powers of this element. The order of the subgroup generated by a (nontrivial) element $a \in G$ is the ***order*** of a. It can be a positive integer or the infinity.

For each subgroup H of a group G, the ***right cosets*** of H are the sets $Ha = \{xa \mid x \in H\}$, $a \in G$. Similarly, the sets aH are the ***left cosets*** of H. The number of distinct right (or left) cosets of H is the ***index*** of H.

26.Ux Lagrange theorem. *If H is a subgroup of a finite group G, then the order of H divides that of G.*

A subgroup H of a group G is ***normal*** if for any $h \in H$ and $a \in G$ we have $aha^{-1} \in H$. Normal subgroups are also called ***normal divisors*** or ***invariant subgroups***.

If the subgroup is normal, then left cosets coincide with right cosets, and the set of cosets is a group with multiplication defined by the formula $(aH)(bH) = abH$. The group of cosets of H in G is called the ***quotient group*** or ***factor group*** of G by H and denoted by G/H.

26.Vx. The kernel $\operatorname{Ker} f$ of a homomorphism $f : G \to H$ is a normal subgroup of G.

26.Wx. *The image $f(G)$ of a homomorphism $f : G \to H$ is isomorphic to the quotient group $G/_{\operatorname{Ker} f}$ of G by the kernel of f.*

26.Xx. The quotient group $\mathbb{R}/_{\mathbb{Z}}$ is canonically isomorphic to the group S^1. Describe the image of the group $\mathbb{Q} \subset \mathbb{R}$ under this isomorphism.

26.Yx. Let G be a group, A a normal subgroup of G, and B an arbitrary subgroup of G. Then AB is also a normal subgroup of G, while $A \cap B$ is a normal subgroup of B. Furthermore, we have $AB/_A \cong B/_{A \cap B}$.

27x. Topological Groups

⌈27′1x⌋ Notion of Topological Group

A *topological group* is a set G equipped with both a topological structure and a group structure such that the maps $G \times G \to G : (x, y) \mapsto xy$ and $G \to G : x \mapsto x^{-1}$ are continuous.

27.1x. Let G be a group and a topological space simultaneously. Prove that the maps $\omega : G \times G \to G : (x, y) \mapsto xy$ and $\alpha : G \to G : x \mapsto x^{-1}$ are continuous iff so is the map $\beta : G \times G \to G : (x, y) \mapsto xy^{-1}$.

27.2x. Prove that if G is a topological group, then the inversion $G \to G : x \mapsto x^{-1}$ is a homeomorphism.

27.3x. Let G be a topological group, X a topological space, $f, g : X \to G$ two maps continuous at a point $x_0 \in X$. Prove that the maps $X \to G : x \mapsto f(x)g(x)$ and $X \to G : x \mapsto (f(x))^{-1}$ are continuous at x_0.

27.Ax. A group equipped with the discrete topology is a topological group.

27.4x. Is a group equipped with the indiscrete topology a topological group?

⌈27′2x⌋ Examples of Topological Groups

27.Bx. The groups listed in *26.Cx* equipped with standard topologies are topological groups.

27.5x. The unit circle $S^1 = \{|z| = 1\} \subset \mathbb{C}$ with the standard multiplication is a topological group.

27.6x. In each of the following situations, check if we have a topological group.

(1) The spaces $\mathbb{R}^n$, $\mathbb{C}^n$, and $\mathbb{H}^n$ with coordinate-wise addition. ($\mathbb{C}^n$ is isomorphic to $\mathbb{R}^{2n}$, while $\mathbb{H}^n$ is isomorphic to $\mathbb{C}^{2n}$.)
(2) The sets $M_n(\mathbb{R})$, $M_n(\mathbb{C})$, and $M_n(\mathbb{H})$ of all $n \times n$ matrices with real, complex, and, respectively, quaternion entries, equipped with the product topology and entry-wise addition. (We identify $M_n(\mathbb{R})$ with $\mathbb{R}^{n^2}$, $M_n(\mathbb{C})$ with $\mathbb{C}^{n^2}$, and $M_n(\mathbb{H})$ with $\mathbb{H}^{n^2}$.)
(3) The sets $GL(n, \mathbb{R})$, $GL(n, \mathbb{C})$, and $GL(n, \mathbb{H})$ of invertible $n \times n$ matrices with real, complex, and quaternionic entries, respectively, under the matrix multiplication.
(4) $SL(n, \mathbb{R})$, $SL(n, \mathbb{C})$, $O(n)$, $O(n, \mathbb{C})$, $U(n)$, $SO(n)$, $SO(n, \mathbb{C})$, $SU(n)$, and other subgroups of $GL(n, K)$ with $K = \mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$.

27.7x. Introduce a topological group structure on the additive group $\mathbb{R}$ that would be distinct from the usual, discrete, and indiscrete topological structures.

27.8x. Find two nonisomorphic connected topological groups that are homeomorphic as topological spaces.

27.9x. On the set $G = [0, 1)$ (equipped with the standard topology), we define addition as follows: $\omega(x, y) = x + y \pmod 1$. Is (G, ω) a topological group?

⌈27′3x⌋ Translations and Conjugations

Let G be a group. Recall that the maps $L_a : G \to G : x \mapsto ax$ and $R_a : G \to G : x \mapsto xa$ are *left* and *right translations through* a, respectively. Note that $L_a \circ L_b = L_{ab}$, while $R_a \circ R_b = R_{ba}$. (To "repair" the last relation, some authors define right translations by $x \mapsto xa^{-1}$.)

27.Cx. *A translation of a topological group is a homeomorphism.*

Recall that the *conjugation* of a group G by an element $a \in G$ is the map $G \to G : x \mapsto axa^{-1}$.

27.Dx. *The conjugation of a topological group by any of its elements is a homeomorphism.*

The following simple observation allows a certain "uniform" treatment of the topology on a group: neighborhoods of distinct points can be compared.

27.Ex. If U is an open set in a topological group G, then for each $x \in G$ the sets xU, Ux, and U^{-1} are open.

27.10x. Does the same hold true for closed sets?

27.11x. Prove that if U and V are subsets of a topological group G and U is open, then UV and VU are open.

27.12x. Will the same hold true if we replace everywhere the word *open* by the word *closed*?

27.13x. Are the following subgroups of the additive group $\mathbb{R}$ closed?

(1) $\mathbb{Z}$,
(2) $\sqrt{2}\,\mathbb{Z}$,
(3) $\mathbb{Z} + \sqrt{2}\,\mathbb{Z}$?

27.14x. Let G be a topological group, $U \subset G$ a compact subset, $V \subset G$ a closed subset. Prove that UV and VU are closed.

27.14x.1. Let F and C be two disjoint subsets of a topological group G. If F is closed and C is compact, then 1_G has a neighborhood V such that $CV \cup VC$ does not meet F. If G is locally compact, then V can be chosen so that $\mathrm{Cl}(CV \cup VC)$ is compact.

⌈27′4x⌋ Neighborhoods

27.Fx. *Let Γ be a neighborhood base of a topological group G at 1_G. Then $\Sigma = \{aU \mid a \in G,\ U \in \Gamma\}$ is a base for topology of G.*

A subset A of a group G is *symmetric* if $A^{-1} = A$.

27.Gx. Any neighborhood of 1 in a topological group contains a symmetric neighborhood of 1.

27.Hx. For any neighborhood U of 1 in a topological group, 1 has a neighborhood V such that $VV \subset U$.

***27.15*x.** Let G be a topological group, U a neighborhood of 1_G, and n a positive integer. Then 1_G has a symmetric neighborhood V such that $V^n \subset U$.

***27.16*x.** Let V be a symmetric neighborhood of 1_G in a topological group G. Then $\bigcup_{n=1}^{\infty} V^n$ is an open-closed subgroup.

***27.17*x.** Let G be a group, Σ be a collection of subsets of G. Prove that G carries a unique topology Ω such that Σ is a neighborhood base for Ω at 1_G and (G, Ω) is a topological group, iff Σ satisfies the following five conditions:

(1) each $U \in \Sigma$ contains 1_G,
(2) for every $x \in U \in \Sigma$, there exists $V \in \Sigma$ such that $xV \subset U$,
(3) for each $U \in \Sigma$, there exists $V \in \Sigma$ such that $V^{-1} \subset U$,
(4) for each $U \in \Sigma$, there exists $V \in \Sigma$ such that $VV \subset U$,
(5) for any $x \in G$ and $U \in \Sigma$, there exists $V \in \Sigma$ such that $V \subset x^{-1}Ux$.

27.I*x. *Riddle. In what sense is *27.Hx* similar to the triangle inequality?

***27.J*x.** Let C be a compact subset of G. Prove that for every neighborhood U of 1_G the unity 1_G has a neighborhood V such that $V \subset xUx^{-1}$ for every $x \in C$.

⌈27′5x⌋ Separation Axioms

***27.K*x.** A topological group G is Hausdorff, iff G satisfies the first separation axiom, iff the unity 1_G (or, more precisely, the singleton $\{1_G\}$) is closed.

***27.L*x.** A topological group G is Hausdorff iff the unity 1_G is the intersection of its neighborhoods.

***27.M*x.** If the unity of a topological group G is closed, then G is regular (as a topological space).

Use the following fact.

27.M*x.*1. Let G be a topological group, $U \subset G$ a neighborhood of 1_G. Then 1_G has a neighborhood V with closure contained in U: $\operatorname{Cl} V \subset U$.

27.N*x *Corollary. *For topological groups, the first three separation axioms are equivalent.*

***27.18*x.** Prove that a finite group carries as many topological group structures as there are normal subgroups. Namely, each finite topological group G contains a normal subgroup N such that the sets gN with $g \in G$ form a base for the topology of G.

⌈27′6x⌋ Countability Axioms

***27.O*x.** If Γ is a neighborhood base at 1_G in a topological group G and $S \subset G$ is a dense set, then $\Sigma = \{aU \mid a \in S, U \in \Gamma\}$ is a base for the topology of G. (Cf. *27.Fx* and *16.H.*)

***27.P*x.** A first countable separable topological group is second countable.

27.19x*. (Cf. *16.Zx*) A first countable Hausdorff topological group G is metrizable. Furthermore, G can be equipped with a right (left) invariant metric.

28x. Constructions

⌈28′1x⌋ Subgroups

***28.A*x.** Let H be a subgroup of a topological group G. Then the topological and group structures induced from G make H a topological group.

***28.1*x.** Let H be a subgroup of an Abelian group G. Prove that, given a structure of topological group in H and a neighborhood base at 1, G carries a structure of topological group with the same neighborhood base at 1.

***28.2*x.** Prove that a subgroup of a topological group is open iff it contains an interior point.

***28.3*x.** Prove that every open subgroup of a topological group is also closed.

***28.4*x.** Prove that every closed subgroup of finite index is also open.

***28.5*x.** Find an example of a subgroup of a topological group that

(1) is closed, but not open;
(2) is neither closed, nor open.

***28.6*x.** Prove that a subgroup H of a topological group is a discrete subspace iff H contains an isolated point.

***28.7*x.** Prove that a subgroup H of a topological group G is closed, iff there exists an open set $U \subset G$ such that $U \cap H = U \cap \operatorname{Cl} H \neq \varnothing$, i.e., iff $H \subset G$ is locally closed at one of its points.

***28.8*x.** Prove that if H is a non-closed subgroup of a topological group G, then $\operatorname{Cl} H \smallsetminus H$ is dense in $\operatorname{Cl} H$.

***28.9*x.** *The closure of a subgroup of a topological group is a subgroup.*

***28.10*x.** Is it true that the interior of a subgroup of a topological group is a subgroup?

***28.B*x.** A connected topological group is generated by any neighborhood of 1.

***28.C*x.** Let H be a subgroup of a group G. Define a relation: $a \sim b$ if $ab^{-1} \in H$. Prove that this is an equivalence relation, and the right cosets of H in G are the equivalence classes.

***28.11*x.** What is the counterpart of *28.C*x for left cosets?

Let G be a topological group, $H \subset G$ a subgroup. The set of left (respectively, right) cosets of H in G is denoted by $G/_H$ (respectively, $_H\backslash G$). The sets $G/_H$ and $_H\backslash G$ carry the quotient topology. Equipped with these topologies, they are called ***spaces of cosets***.

***28.D*x.** For any topological group G and its subgroup H, the natural projections $G \to G/_H$ and $G \to {}_H\backslash G$ are open (i.e., the image of every open set is open).

***28.E*x.** The space of left (or right) cosets of a closed subgroup in a topological group is regular.

***28.F*x.** The group G is compact (respectively, connected) if so are H and $G/_H$.

***28.12*x.** If H is a connected subgroup of a group G, then the preimage of each connected component of $G/_H$ is a connected component of G.

***28.13*x.** We regard the group $SO(n-1)$ as a subgroup of $SO(n)$. If $n \geq 2$, then the space $SO(n)/_{SO(n-1)}$ is homeomorphic to S^{n-1}.

***28.14*x.** The groups $SO(n)$, $U(n)$, $SU(n)$, and $Sp(n)$ are 1) compact and 2) connected for any $n \geq 1$. 3) How many connected components do the groups $O(n)$ and $O(p,q)$ have? (Here, $O(p,q)$ is the group of linear transformations in $\mathbb{R}^{p+q}$ preserving the quadratic form $x_1^2 + \dots + x_p^2 - y_1^2 - \dots - y_q^2$.)

⌈28′2x⌋ Normal Subgroups

***28.G*x.** Prove that the closure of a normal subgroup of a topological group is a normal subgroup.

***28.H*x.** The connected component of 1 in a topological group is a closed normal subgroup.

***28.15*x.** The path-connected component of 1 in a topological group is a normal subgroup.

***28.I*x.** The quotient group of a topological group is a topological group (provided that it is equipped with the quotient topology).

***28.J*x.** The natural projection of a topological group onto its quotient group is open.

***28.K*x.** If a topological group G is first (respectively, second) countable, then so is any quotient group of G.

***28.L*x.** Let H be a normal subgroup of a topological group G. Then the quotient group G/H is regular iff H is closed.

***28.M*x.** Prove that a normal subgroup H of a topological group G is open iff the quotient group G/H is discrete.

The *center* of a group G is the set $C(G) = \{x \in G \mid xg = gx \text{ for each } g \in G\}$.

***28.16*x.** Each discrete normal subgroup H of a connected group G is contained in the center of G.

⌈28′3x⌋ Homomorphisms

In the case of topological groups, a ***homomorphism*** is a *continuous* group homomorphism.

28.Nx. *Let G and H be two topological groups. A group homomorphism $f : G \to H$ is continuous iff f is continuous at 1_G.*

Not counting similar modifications, which can be summarized by the following principle: *everything is assumed to respect the topological structures*, the terminology of group theory carries over without changes. In particular, an ***isomorphism*** in group theory is an invertible homomorphism. Its inverse is a homomorphism (and hence an isomorphism) automatically. In the theory of topological groups, this must be included in the definition: an ***isomorphism*** of topological groups is an invertible homomorphism whose inverse is also a homomorphism. In other words, an isomorphism of topological groups is a map that is both a group isomorphism and a homeomorphism. Cf. Section 11.

28.17x. Prove that the map $[0, 1) \to S^1 : \ x \mapsto e^{2\pi i x}$ is a topological group homomorphism.

28.Ox. An epimorphism $f : G \to H$ is an open map iff the injective factor $f/S(f) : G/\operatorname{Ker} f \to H$ of f is an isomorphism.

28.Px. An epimorphism of a compact topological group onto a topological group with closed unity is open.

28.Qx. Prove that the quotient group $\mathbb{R}/\mathbb{Z}$ of the additive group $\mathbb{R}$ by the subgroup $\mathbb{Z}$ is isomorphic to the multiplicative group $S^1 = \{z \in \mathbb{C} : |z| = 1\}$ of complex numbers with absolute value 1.

⌈28′4x⌋ Local Isomorphisms

Let G and H be two topological groups. A ***local isomorphism*** from G to H is a homeomorphism f of a neighborhood U of 1_G in G onto a neighborhood V of 1_H in H such that

- $f(xy) = f(x)f(y)$ for any $x, y \in U$ such that $xy \in U$,
- $f^{-1}(zt) = f^{-1}(z)f^{-1}(t)$ for any $z, t \in V$ such that $zt \in V$.

Two topological groups G and H are ***locally isomorphic*** if there exists a local isomorphism from G to H.

28.Rx. Isomorphic topological groups are locally isomorphic.

28.Sx. The additive group $\mathbb{R}$ and the multiplicative group $S^1 \subset \mathbb{C}$ are locally isomorphic, but not isomorphic.

28.18x. Prove that local isomorphism of topological groups is an equivalence relation.

28.19x. Find neighborhoods of unities in $\mathbb{R}$ and S^1 and a homeomorphism between them that satisfies the first condition in the definition of local isomorphism, but does not satisfy the second one.

28.20x. Prove that if a homeomorphism between neighborhoods of unities in two topological groups satisfies only the first condition in the definition of local isomorphism, then it has a submap that is a local isomorphism between these topological groups.

⌈28′5x⌋ Direct Products

Let G and H be two topological groups. In group theory, the product $G \times H$ is given a group structure.[1] In topology, it is given a topological structure (see Section 20).

28.Tx. These two structures are compatible: the group operations in $G \times H$ are continuous with respect to the product topology.

Thus, $G \times H$ is a topological group. It is called the *direct product* of the topological groups G and H. There are canonical homomorphisms related to this: the inclusions $i_G : G \to G \times H : x \mapsto (x, 1)$ and $i_H : H \to G \times H : x \mapsto (1, x)$, which are monomorphisms, and the projections $\mathrm{pr}_G : G \times H \to G : (x, y) \mapsto x$ and $\mathrm{pr}_H : G \times H \to H : (x, y) \mapsto y$, which are epimorphisms.

28.21x. Prove that the topological groups $(G \times H)/_{i_H(H)}$ and G are isomorphic.

28.22x. The product operation is both commutative and associative: $G \times H$ is (canonically) isomorphic to $H \times G$, while $G \times (H \times K)$ is canonically isomorphic to $(G \times H) \times K$.

A topological group G *decomposes into a direct product of two subgroups* A *and* B if the map $A \times B \to G : (x, y) \mapsto xy$ is a topological group isomorphism. If this is the case, then the groups G and $A \times B$ are usually identified via this isomorphism.

Recall that a similar definition exists in ordinary group theory. The only difference is that in ordinary group theory an isomorphism is just an algebraic isomorphism. Furthermore, in that theory, G decomposes into a direct product of its subgroups A and B iff A and B generate G, A and B are normal subgroups, and $A \cap B = \{1\}$. Therefore, if these conditions are fulfilled in the case of topological groups, then $A \times B \to G : (x, y) \mapsto xy$ is a group isomorphism.

28.23x. Prove that in this situation the map $A \times B \to G : (x, y) \mapsto xy$ is continuous. Find an example where the inverse group isomorphism is not continuous.

[1] Recall that the multiplication in $G \times H$ is defined by the formula $(x, u)(y, v) = (xy, uv)$.

***28.U*x.** Prove that if a compact Hausdorff group G decomposes algebraically into a direct product of two closed subgroups, then G also decomposes into a direct product of these subgroups as a topological group.

***28.24*x.** Prove that the multiplicative group $\mathbb{R} \smallsetminus 0$ of nonzero reals is isomorphic (as a topological group) to the direct product of the multiplicative groups $S^0 = \{1, -1\}$ and $\mathbb{R}_{>0} = \{x \in \mathbb{R} \mid x > 0\}$.

***28.25*x.** Prove that the multiplicative group $\mathbb{C} \smallsetminus 0$ of nonzero complex numbers is isomorphic (as a topological group) to the direct product of the multiplicative groups $S^1 = \{z \in \mathbb{C} : |z| = 1\}$ and $\mathbb{R}_{>0}$.

***28.26*x.** Prove that the multiplicative group $\mathbb{H} \smallsetminus 0$ of nonzero quaternions is isomorphic (as a topological group) to the direct product of the multiplicative groups $S^3 = \{z \in \mathbb{H} : |z| = 1\}$ and $\mathbb{R}_{>0}$.

***28.27*x.** Prove that the subgroup $S^0 = \{1, -1\}$ of $S^3 = \{z \in \mathbb{H} : |z| = 1\}$ is not a direct factor.

***28.28*x.** Find a topological group homeomorphic to $\mathbb{R}P^3$ (the three-dimensional real projective space).

Let a group G contain a normal subgroup A and a subgroup B such that $AB = G$ and $A \cap B = \{1_G\}$. If B is also normal, then G is the direct product $A \times B$. Otherwise, G is a ***semidirect product*** of A and B.

***28.V*x.** Let a topological group G be a semidirect product of its subgroups A and B. If for any neighborhoods of unity, $U \subset A$ and $V \subset B$, their product UV contains a neighborhood of 1_G, then G is homeomorphic to $A \times B$.

⌈28′6x⌋ Groups of Homeomorphisms

For any topological space X, the autohomeomorphisms of X form a group under composition as the group operation. We denote this group by $\operatorname{Top} X$. To make this group topological, we slightly enlarge the topological structure induced on $\operatorname{Top} X$ by the compact-open topology of $\mathcal{C}(X, X)$.

***28.W*x.** The collection of the sets $W(C, U)$ and $(W(C, U))^{-1}$ taken over all compact $C \subset X$ and open $U \subset X$ is a subbase for the topological structure on $\operatorname{Top} X$.

In what follows, we equip $\operatorname{Top} X$ with this topological structure.

***28.X*x.** If X is Hausdorff and locally compact, then $\operatorname{Top} X$ is a topological group.

28.X*x*.1. If X is Hausdorff and locally compact, then the map $\operatorname{Top} X \times \operatorname{Top} X \to \operatorname{Top} X : (g, h) \mapsto g \circ h$ is continuous.

29x. Actions of Topological Groups

⌈29′1x⌋ Action of a Group on a Set

A *left action* of a group G on a set X is a map $G \times X \to X : (g, x) \mapsto gx$ such that $1x = x$ for each $x \in X$ and $(gh)x = g(hx)$ for each $x \in X$ and any $g, h \in G$. A set X equipped with such an action is a *left G-set*. Right G-sets are defined in a similar way.

29.Ax. If X is a left G-set, then $G \times X \to X : (x, g) \mapsto g^{-1}x$ is a right action of G on X.

29.Bx. If X is a left G-set, then the map $X \to X : x \mapsto gx$ is a bijection for each $g \in G$.

A left action of G on X is *effective* (or *faithful*) if for each $g \in G \smallsetminus 1$ the map $G \to G : x \mapsto gx$ is not equal to id_G. Let X_1 and X_2 be two left G-sets. A map $f : X_1 \to X_2$ is *G-equivariant* if $f(gx) = gf(x)$ for any $x \in X$ and $g \in G$.

We say that X is a *homogeneous left G-set*, or, rather, that G acts on X *transitively* if there exists $g \in G$ such that $y = gx$ for any $x, y \in X$.

The same terminology applies to right actions with obvious modifications.

29.Cx. The natural actions of G on $G/_H$ and $_H\backslash G$ transform $G/_H$ and $_H\backslash G$ into homogeneous left and, respectively, right G-sets.

Let X be a homogeneous left G-set. Consider a point $x \in X$ and the set $G^x = \{g \in G \mid gx = x\}$. We easily see that G^x is a subgroup of G. It is called the *isotropy subgroup* of x.

29.Dx. Each homogeneous left (respectively, right) G-set X is isomorphic to $G/_H$ (respectively, $_H\backslash G$), where H is the isotropy group of a certain point in X.

29.Dx.1. All isotropy subgroups G^x, $x \in X$, are pairwise conjugate.

Recall that the *normalizer* $Nr(H)$ of a subgroup H of a group G consists of all elements $g \in G$ such that $gHg^{-1} = H$. This is the largest subgroup of G containing H as a normal subgroup.

29.Ex. The group of all automorphisms of a homogeneous G-set X is isomorphic to $N(H)/_H$, where H is the isotropy group of a certain point in X.

29.Ex.1. If two points $x, y \in X$ have the same isotropy group, then X has an automorphism sending x to y.

⌈29′2x⌋ Continuous Action

We speak about a *left G-space* X if X is a topological space, G is a topological group acting on X, and the action $G \times X \to X$ is continuous (as a map). All terminology (and definitions) concerning G-sets extends to G-spaces literally.

Note that if G is a discrete group, then each action of G by homeomorphisms is continuous and thus provides a G-space.

29.Fx. Let X be a left G-space. Then the natural map $\phi : G \to \operatorname{Top} X$ induced by this action is a group homomorphism.

29.Gx. If in the assumptions of Problem *29.Fx* the G-space X is Hausdorff and locally compact, then the induced homomorphism $\phi : G \to \operatorname{Top} X$ is continuous.

29.1x. In each of the following situations, check if we have a continuous action and a continuous homomorphism $G \to \operatorname{Top} X$:

(1) G is a topological group, $X = G$, and G acts on X by left (or right) translations, or by conjugation;
(2) G is a topological group, $H \subset G$ is a subgroup, $X = G/_H$, and G acts on X via $g(aH) = (ga)H$;
(3) $G = GL(n, K)$ (where $K = \mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$)), and G acts on K^n via matrix multiplication;
(4) $G = GL(n, K)$ (where $K = \mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$), and G acts on KP^{n-1} via matrix multiplication;
(5) $G = O(n, \mathbb{R})$, and G acts on S^{n-1} via matrix multiplication;
(6) the (additive) group $\mathbb{R}$ acts on the torus $S^1 \times \cdots \times S^1$ according to formula $(t, (w_1, \dots, w_r)) \mapsto (e^{2\pi i a_1 t} w_1, \dots, e^{2\pi i a_r t} w_r)$; this action is an *irrational flow* if $a_1, \dots, a_r$ are linearly independent over $\mathbb{Q}$.

If the action of G on X is not effective, then we can consider its kernel

$$G^{\mathrm{Ker}} = \{g \in G \mid gx = x \ \text{ for all } \ x \in X\}.$$

This kernel is a closed normal subgroup of G, and the topological group $G/_{G^{\mathrm{Ker}}}$ acts naturally and effectively on X.

29.Hx. The formula $gG^{\mathrm{Ker}}(x) = gx$ determines an effective continuous action of $G/_{G^{\mathrm{Ker}}}$ on X.

A group G acts *properly discontinuously* on X if for each compact set $C \subset X$ the set $\{g \in G \mid (gC) \cap C \neq \varnothing\}$ is finite.

29.Ix. If G acts properly discontinuously and effectively on a Hausdorff locally compact space X, then $\phi(G)$ is a discrete subset of $\operatorname{Top} X$. (Here, as before, $\phi : G \to \operatorname{Top} X$ is the monomorphism induced by the G-action.) In particular, G is a discrete group.

29.2x. List, up to similarity, all triangles $T \subset \mathbb{R}^2$ such that the reflections in the sides of T generate a group acting on $\mathbb{R}^2$ properly discontinuously.

⌈29′3x⌋ Orbit Spaces

Let X be a left G-space. For $x \in X$, the set $G(x) = \{gx \mid g \in G\}$ is the *orbit* of x. In terms of orbits, the action of G on X is transitive iff it has only one orbit. For $A \subset X$ and $E \subset G$, we put $E(A) = \{ga \mid g \in E,\ a \in A\}$. We denote the set of all orbits by X/G and equip it with the quotient topology.

29.Jx. *Let G be a compact topological group acting on a Hausdorff space X. Then the canonical map $G/_{G^x} \to G(x)$ is a homeomorphism for each $x \in X$.*

29.3x. Give an example where X is Hausdorff, but $G/_{G_x}$ is not homeomorphic to $G(x)$.

29.Kx. If a compact topological group G acts on a compact Hausdorff space X, then $X/_G$ is a compact Hausdorff space.

29.4x. Let G be a compact group, X a Hausdorff G-space, $A \subset X$. If A is closed (respectively, compact), then so is $G(A)$.

29.5x. Consider the canonical action of $G = \mathbb{R} \smallsetminus 0$ on $X = \mathbb{R}$ (by multiplication). Find all orbits and all isotropy subgroups of this action. Recognize $X/_G$ as a topological space.

29.6x. Let G be the group generated by reflections in the sides of a rectangle in $\mathbb{R}^2$. Recognize the quotient space $\mathbb{R}^2/_G$ as a topological space. Recognize the group G.

29.7x. Let G be the group from Problem *29.6x*, and let $H \subset G$ be the subgroup of index 2 constituted by the orientation-preserving elements in G. Recognize the quotient space $\mathbb{R}^2/_H$ as a topological space. Recognize the groups G and H.

29.8x. Consider the following (diagonal) action of the torus $G = (S^1)^{n+1}$ on $X = \mathbb{C}P^n$: $(z_0{:}z_1{:}\dots{:}z_n) \mapsto (\theta_0 z_0{:}\theta_1 z_1{:}\dots{:}\theta_n z_n)$. Find all orbits and isotropy subgroups. Recognize $X/_G$ as a topological space.

29.9x. Consider the canonical action (by permutations of coordinates) of the symmetric group $G = \mathbb{S}_n$ on $X = \mathbb{R}^n$ and $X = \mathbb{C}^n$, respectively. Recognize $X/_G$ as a topological space.

29.10x. Let $G = SO(3)$ act on the space X of symmetric 3×3 real matrices with trace 0 by conjugations $x \mapsto gxg^{-1}$. Recognize $X/_G$ as a topological space. Find all orbits and isotropy groups.

⌈29′4x⌋ Homogeneous Spaces

A G-space is *homogeneous* if the action of G is transitive.

29.Lx. Let G be a topological group, $H \subset G$ a subgroup. Then G is a homogeneous H-space under the translation action of H. The quotient space $G/_H$ is a homogeneous G-space under the induced action of G.

29.Mx. *Let X be a Hausdorff homogeneous G-space. If X and G are locally compact and G is second countable, then X is homeomorphic to $G/_{G^x}$ for each $x \in X$.*

29.Nx. Let X be a homogeneous G-space. Then the canonical map $G/G^x \to X$, $x \in X$, is a homeomorphism iff it is open.

29.11x. Show that $O(n+1)/O(n) = S^n$ and $U(n)/U(n-1) = S^{2n-1}$.

29.12x. Show that $O(n+1)/O(n) \times O(1) = \mathbb{R}P^n$ and $U(n)/U(n-1) \times U(1) = \mathbb{C}P^n$.

29.13x. Show that $Sp(n)/Sp(n-1) = S^{4n-1}$, where

$$Sp(n) = \{A \in GL(\mathbb{H}) \mid AA^* = I\}.$$

29.14x. Represent the torus $S^1 \times S^1$ and the Klein bottle as homogeneous spaces.

29.15x. Give a geometric interpretation of the following homogeneous spaces: 1) $O(n)/O(1)^n$, 2) $O(n)/O(k) \times O(n-k)$, 3) $O(n)/SO(k) \times O(n-k)$, and 4) $O(n)/O(k)$.

29.16x. Represent $S^2 \times S^2$ as a homogeneous space.

29.17x. Recognize $SO(n,1)/SO(n)$ as a topological space.

Part 2

Elements of Algebraic Topology

This part of the book can be considered an introduction to algebraic topology, which is a part of topology that relates topological and algebraic problems. The relationship is used in both directions, but the reduction of topological problems to algebra is more useful at first stages because algebra is usually easier.

The relation is established according to the following scheme. One invents a construction that assigns to each topological space X under consideration an algebraic object $A(X)$. The latter may be a group, a ring, a space with a quadratic form, an algebra, etc. Another construction assigns to a continuous map $f : X \to Y$ a homomorphism $A(f) : A(X) \to A(Y)$. The constructions satisfy natural conditions (in particular, they form a functor), which make it possible to relate topological phenomena with their algebraic images obtained via the constructions.

There is an immense number of useful constructions of this kind. In this part, we deal mostly with one of them which, historically, was the first one: the fundamental group of a topological space. It was invented by Henri Poincaré in the end of the XIXth century.

Chapter VI

Fundamental Group

30. Homotopy

⌈30′1⌋ Continuous Deformations of Maps

30.A. Is it possible to deform continuously:

(1) the identity map $\mathrm{id} : \mathbb{R}^2 \to \mathbb{R}^2$ into the constant map $\mathbb{R}^2 \to \mathbb{R}^2 : x \mapsto 0$,

(2) the identity map $\mathrm{id} : S^1 \to S^1$ into the symmetry $S^1 \to S^1 : x \mapsto -x$ (here x is considered a complex number because the circle S^1 is $\{x \in \mathbb{C} : |x| = 1\}$),

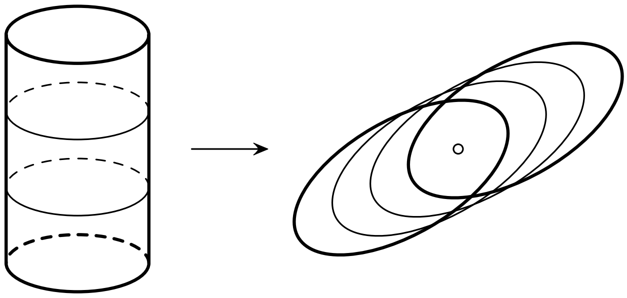
(3) the identity map $\mathrm{id} : S^1 \to S^1$ into the constant map $S^1 \to S^1 : x \mapsto 1$,

(4) the identity map $\mathrm{id} : S^1 \to S^1$ into the two-fold wrapping $S^1 \to S^1 : x \mapsto x^2$,

(5) the inclusion $S^1 \to \mathbb{R}^2$ into a constant map,

(6) the inclusion $S^1 \to \mathbb{R}^2 \smallsetminus 0$ into a constant map?

30.B. Riddle. When you (tried to) solve the previous problem, what did you mean by "*deform continuously*"?



The present section is devoted to the notion of *homotopy* formalizing the naive idea of continuous deformation of a map.

⌈30′2⌋ Homotopy as a Map and a Family of Maps

Let f and g be two continuous maps of a topological space X to a topological space Y, and let $H : X \times I \to Y$ be a continuous map such that $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$ for each $x \in X$. Then f and g are *homotopic*, and H is a *homotopy* between f and g.

For $x \in X$ and $t \in I$, we denote $H(x, t)$ by $h_t(x)$. This change of notation results in a change of the point of view of H. Indeed, for a fixed t the formula $x \mapsto h_t(x)$ determines a map $h_t : X \to Y$, and H becomes a family of maps h_t enumerated by $t \in I$.

30.C. Each h_t is continuous.

30.D. Does continuity of all h_t imply continuity of H?

The conditions $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$ in the above definition of a homotopy can be reformulated as follows: $h_0 = f$ and $h_1 = g$. Thus, a homotopy between f and g can be regarded as a family of continuous maps that connects f and g. Continuity of a homotopy allows us to say that it is a *continuous family of continuous maps* (see 30′10).

⌈30′3⌋ Homotopy as a Relation

30.E. *Homotopy of maps is an equivalence relation.*

30.E.1. *If $f : X \to Y$ is a continuous map, then $H : X \times I \to Y : (x, t) \mapsto f(x)$ is a homotopy between f and f.*

30.E.2. *If H is a homotopy between f and g, then H' defined by $H'(x, t) = H(x, 1 - t)$ is a homotopy between g and f.*

30.E.3. *If H is a homotopy between f and f' and H' is a homotopy between f' and f'', then H'' defined by*

$$H''(x,t) = \begin{cases} H(x, 2t) & \text{if } t \in [0, 1/2], \\ H'(x, 2t - 1) & \text{if } t \in [1/2, 1] \end{cases}$$

is a homotopy between f and f''.

Homotopy, being an equivalence relation by *30.E*, splits the set $\mathcal{C}(X,Y)$ of all continuous maps from a space X to a space Y into equivalence classes. The latter are *homotopy classes*. The set of homotopy classes of all continuous maps $X \to Y$ is denoted by $\pi(X,Y)$. Maps homotopic to a constant map are also said to be *null-homotopic* .

30.1. Prove that the set $\pi(X,I)$ is a singleton for each X.

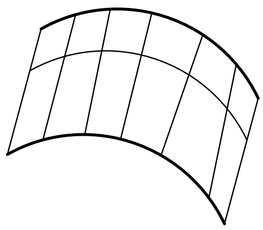
30.2. Prove that two constant maps $X \to Y$ are homotopic iff their images lie in one path-connected component of Y.

30.3. Prove that the number of elements of $\pi(I,Y)$ is equal to the number of path-connected components of Y.

⌈30′4⌋ Rectilinear Homotopy

30.F. Any two continuous maps of the same space to $\mathbb{R}^n$ are homotopic.

30.G. Solve the preceding problem by proving that for continuous maps $f,g : X \to \mathbb{R}^n$, the formula $H(x,t) = (1-t)f(x) + tg(x)$ determines a homotopy between f and g.



The homotopy defined in *30.G* is a *rectilinear* homotopy.

30.H. Any two continuous maps of an arbitrary space to a convex subspace of $\mathbb{R}^n$ are homotopic.

⌈30′5⌋ Maps to Star-Shaped Sets

A set $A \subset \mathbb{R}^n$ is *star-shaped* if A contains a point a such that for any $x \in A$ the whole segment $[a,x]$ connecting x to a is contained in A. The point a is the *center* of the star. (Certainly, the center of the star is not uniquely determined.)

30.4. Prove that any two continuous maps of a space to a star-shaped subspace of $\mathbb{R}^n$ are homotopic.

⌈30′6⌋ Maps of Star-Shaped Sets

30.5. Prove that any continuous map of a star-shaped set $C \subset \mathbb{R}^n$ to any space is null-homotopic.

30.6. Under what conditions (formulated in terms of known topological properties of a space X) are any two continuous maps of any star-shaped set to X homotopic?

⌈30′7⌋ Easy Homotopies

30.7. Prove that each non-surjective map of any topological space to S^n is null-homotopic.

30.8. Prove that any two maps of a one-point space to $\mathbb{R}^n \smallsetminus 0$ with $n > 1$ are homotopic.

30.9. Find two nonhomotopic maps from a one-point space to $\mathbb{R} \smallsetminus 0$.

30.10. For various m, n, and k, calculate the number of homotopy classes of maps $\{1, 2, \dots, m\} \to \mathbb{R}^n \smallsetminus \{x_1, x_2, \dots, x_k\}$, where $\{1, 2, \dots, m\}$ is equipped with discrete topology.

30.11. Let f and g be two maps from a topological space X to $\mathbb{C} \smallsetminus 0$. Prove that if $|f(x) - g(x)| < |f(x)|$ for any $x \in X$, then f and g are homotopic.

30.12. Prove that for any polynomials p and q over $\mathbb{C}$ of the same degree in one variable there exists $r > 0$ such that for any $R > r$ the formulas $z \mapsto p(z)$ and $z \mapsto q(z)$ determine maps of the circle $\{z \in \mathbb{C} : |z| = R\}$ to $\mathbb{C} \smallsetminus 0$ and these maps are homotopic.

30.13. Let f and g be two maps of an arbitrary topological space X to S^n. Prove that if $|f(a) - g(a)| < 2$ for each $a \in X$, then f is homotopic to g.

30.14. Let $f : S^n \to S^n$ be a continuous map. Prove that if it is fixed-point-free, i.e., $f(x) \neq x$ for every $x \in S^n$, then f is homotopic to the symmetry $x \mapsto -x$.

⌈30′8⌋ Two Natural Properties of Homotopies

30.I. Let $f, f' : X \to Y$, $g : Y \to B$, and $h : A \to X$ be continuous maps, and let $F : X \times I \to Y$ be a homotopy between f and f'. Prove that then $g \circ F \circ (h \times \mathrm{id}_I)$ is a homotopy between $g \circ f \circ h$ and $g \circ f' \circ h$.

30.J. Riddle. In the assumptions of *30.I*, define a natural map

$$\pi(X, Y) \to \pi(A, B).$$

How does it depend on g and h? Write down all nice properties of this construction.

30.K. Prove that two maps $f_0, f_1 : X \to Y \times Z$ are homotopic iff $\mathrm{pr}_Y \circ f_0$ is homotopic to $\mathrm{pr}_Y \circ f_1$ and $\mathrm{pr}_Z \circ f_0$ is homotopic to $\mathrm{pr}_Z \circ f_1$.

⌈30′9⌋ Stationary Homotopy

Let A be a subset of X. A homotopy $H : X \times I \to Y$ is *fixed* or *stationary* on A, or, briefly, an A-*homotopy* if $H(x, t) = H(x, 0)$ for all $x \in A$, $t \in I$. Two maps connected by an A-homotopy are A-*homotopic*.

Certainly, any two A-homotopic maps coincide on A. If we want to emphasize that a homotopy is not assumed to be fixed, then we say that it is *free*. If we want to emphasize the opposite (that the homotopy is fixed), then we say that it is *relative*.[1]

[1] Warning: there is a similar, but different kind of homotopy, which is also called relative.

30.L. Prove that, like free homotopy, A-homotopy is an equivalence relation.

The classes into which the A-homotopy splits the set of continuous maps $X \to Y$ that agree on A with a map $f : A \to Y$ are *A-homotopy classes of continuous extensions of f to X*.

30.M. For what A is a rectilinear homotopy fixed on A?

⌈30′10⌋ Homotopies and Paths

Recall that a *path* in a space X is a continuous map from the segment I to X. (See Section 14.)

30.N. Riddle. In what sense is any path a homotopy?

30.O. Riddle. In what sense does any homotopy consist of paths?

30.P. Riddle. In what sense is any homotopy a path?

Recall that the *compact-open topology* in $\mathcal{C}(X, Y)$ is the topology generated by the sets $\{\varphi \in \mathcal{C}(X, Y) \mid \varphi(A) \subset B\}$ for compact $A \subset X$ and open $B \subset Y$.

30.15. Prove that any homotopy $h_t : X \to Y$ determines (see 30′2) a path in $\mathcal{C}(X, Y)$ with compact-open topology.

30.16. Prove that if X is locally compact and regular, then any path in $\mathcal{C}(X, Y)$ with compact-open topology determines a homotopy.

⌈30′11⌋ Homotopy of Paths

30.Q. Prove that two paths in a space X are freely homotopic iff their images belong to the same path-connected component of X.

This shows that the notion of free homotopy in the case of paths is not interesting. On the other hand, there is a sort of relative homotopy playing a very important role. This is $(0 \cup 1)$-homotopy. This causes the following commonly accepted deviation from the terminology introduced above: homotopy of paths always means not a free homotopy, but a homotopy fixed on the endpoints of I (i.e., on $0 \cup 1$).

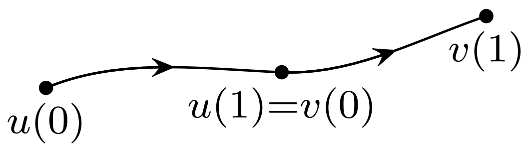
Notation: a homotopy class of a path s is denoted by $[s]$.

31. Homotopy Properties of Path Multiplication

⌈31′1⌋ Multiplication of Homotopy Classes of Paths

Recall (see Section 14) that two paths u and v in a space X can be multiplied, provided that the initial point $v(0)$ of v is the final point $u(1)$ of u. The product uv is defined by

$$uv(t) = \begin{cases} u(2t) & \text{if } t \in [0, 1/2], \\ v(2t-1) & \text{if } t \in [1/2, 1]. \end{cases}$$



31.A. *If a path u is homotopic to u', a path v is homotopic to v', and the product uv exists, then $u'v'$ exists and is homotopic to uv.*

Define the product of homotopy classes of paths u and v as the homotopy class of uv. So, $[u][v]$ is defined as $[uv]$, provided that uv is defined. This is a definition requiring a proof.

31.B. *The product of homotopy classes of paths is well defined.*[2]

⌈31′2⌋ Associativity

31.C. Is multiplication of paths associative?

Certainly, this question might be formulated in more detail as follows.

31.D. Let u, v, and w be paths in a certain space such that products uv and vw are defined (i.e., $u(1) = v(0)$ and $v(1) = w(0)$). Is it true that $(uv)w = u(vw)$?

31.1. Prove that for paths in a metric space $(uv)w = u(vw)$ implies that u, v, and w are constant maps.

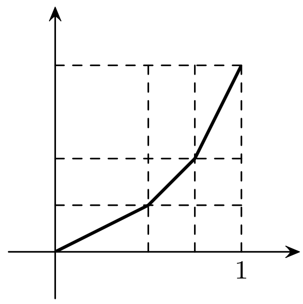
31.2. Riddle. Find nonconstant paths u, v, and w in an indiscrete space such that $(uv)w = u(vw)$.

31.E. *Multiplication of homotopy classes of paths is associative.*

[2] Of course, when the initial point of paths in the first class is the final point of paths in the second class.

31.E.1. Reformulate Theorem *31.E* in terms of paths and their homotopies.

31.E.2. Find a map $\varphi : I \to I$ such that if u, v, and w are paths with $u(1) = v(0)$ and $v(1) = w(0)$, then $((uv)w) \circ \varphi = u(vw)$.



31.E.3. Any path in I starting at 0 and ending at 1 is homotopic to $\mathrm{id} : I \to I$.

31.E.4. Let u, v, and w be paths in a space such that products uv and vw are defined (thus, $u(1) = v(0)$ and $v(1) = w(0)$). Then $(uv)w$ is homotopic to $u(vw)$.

If you want to understand the essence of *31.E*, then observe that the paths $(uv)w$ and $u(vw)$ have the same trajectories and differ only by the time spent in different fragments of the path. Therefore, in order to find a homotopy between them, we must find a continuous way to change one schedule to the other. The lemmas above suggest a formal way of such a change, but the same effect can be achieved in many other ways.

31.3. Present explicit formulas for the homotopy H between the paths $(uv)w$ and $u(vw)$.

⌈31′3⌋ Unit

Let a be a point of a space X. Denote by e_a the path $I \to X : t \mapsto a$.

31.F. Is e_a a unit for multiplication of paths?

The same question in more detailed form:

31.G. Is $e_a u = u$ for paths u with $u(0) = a$? Is $ve_a = v$ for paths v with $v(1) = a$?

31.4. Prove that if $e_a u = u$ and the space satisfies the first separation axiom, then $u = e_a$.

31.H. *The homotopy class of e_a is a unit for multiplication of homotopy classes of paths.*

⌈31′4⌋ Inverse

Recall that a path u has the inverse path $u^{-1} : t \mapsto u(1-t)$ (see Section 14).

31.I. Is the inverse path inverse with respect to multiplication of paths?

In other words:

31.J. For a path u beginning in a and finishing in b, is it true that $uu^{-1} = e_a$ and $u^{-1}u = e_b$?

31.5. Prove that for a path u with $u(0) = a$ equality $uu^{-1} = e_a$ implies $u = e_a$.

31.K. *For any path u, the homotopy class of the path u^{-1} is inverse to the homotopy class of u.*

31.K.1. Find a map $\varphi : I \to I$ such that $uu^{-1} = u \circ \varphi$ for any path u.

31.K.2. Any path in I that starts and finishes at 0 is homotopic to the constant path $e_0 : I \to I$.

We see that from the algebraic point of view multiplication of paths is terrible, but it determines multiplication of homotopy classes of paths, which has nice algebraic properties. The only unfortunate property is that the multiplication of homotopy classes of paths is defined not for any two classes.

31.L. Riddle. How to select a subset of the set of homotopy classes of paths to obtain a group?

32. Fundamental Group

⌈32′1⌋ Definition of Fundamental Group

Let X be a topological space, x_0 a point of X. A path in X which starts and ends at x_0 is a *loop* in X at x_0. Denote by $\Omega_1(X, x_0)$ the set of loops in X at x_0. Denote by $\pi_1(X, x_0)$ the set of homotopy classes of loops in X at x_0.

Both $\Omega_1(X, x_0)$ and $\pi_1(X, x_0)$ are equipped with a multiplication.

32.A. *For any topological space X and a point $x_0 \in X$, the set $\pi_1(X, x_0)$ of homotopy classes of loops at x_0 with multiplication defined above in Section 31 is a group.*

$\pi_1(X, x_0)$ is the *fundamental group* of the space X with base point x_0. It was introduced by Poincaré, and this is why it is also called the *Poincaré group*. The letter π in this notation is also due to Poincaré.

⌈32′2⌋ Why Index 1?

The index 1 in the designation $\pi_1(X, x_0)$ appeared later than the letter π. It is related to one more name of the fundamental group: the first (or one-dimensional) homotopy group. There is an infinite sequence of groups $\pi_r(X, x_0)$ with $r = 1, 2, 3, \dots$, the fundamental group being one of them. The higher-dimensional homotopy groups were defined by Witold Hurewicz in 1935, thirty years after the fundamental group was defined. Roughly speaking, the general definition of $\pi_r(X, x_0)$ is obtained from the definition of $\pi_1(X, x_0)$ by replacing I with the cube I^r.

32.B. Riddle. How to generalize problems of this section in such a way that in each of them I would be replaced by I^r?

There is even a "zero-dimensional homotopy group" $\pi_0(X, x_0)$, but it is not a group, as a rule. It is the set of path-connected components of X. Although there is no natural multiplication in $\pi_0(X, x_0)$, unless X is equipped with some special additional structures, $\pi_0(X, x_0)$ has a natural unit. This is the component containing x_0.

⌈32′3⌋ Circular loops

Let X be a topological space, $x_0 \in X$. A continuous map $l : S^1 \to X$ such that[3] $l(1) = x_0$ is a (*circular*) *loop* at x_0. Assign to each circular loop l the composition of l with the exponential map $I \to S^1 : t \mapsto e^{2\pi i t}$. This is a usual loop at the same point.

[3]Recall that S^1 is regarded as a subset of the plane R^2, and the latter is identified with $\mathbb{C}$ in a canonical way. Hence, $1 \in S^1 = \{z \in \mathbb{C} : |z| = 1\}$.

32.C. Prove that any loop is obtained in this way from a circular loop.

Two circular loops l_1 and l_2 are ***homotopic*** if they are 1-homotopic. A homotopy of a circular loop not fixed at x_0 is a ***free*** homotopy.

32.D. Prove that two circular loops are homotopic iff the corresponding ordinary loops are homotopic.

32.1. What kind of homotopy of loops corresponds to free homotopy of circular loops?

32.2. Describe the operation with circular loops corresponding to the multiplication of paths.

32.3. Let U and V be the circular loops with common base point $U(1) = V(1)$ corresponding to the loops u and v. Prove that the circular loop

$$z \mapsto \begin{cases} U(z^2) \text{ if } \operatorname{Im}(z) \geq 0, \\ V(z^2) \text{ if } \operatorname{Im}(z) \leq 0 \end{cases}$$

corresponds to the product of u and v.

32.4. Outline a construction of fundamental group using circular loops.

⌈32′4⌋ The Very First Calculations

32.E. Prove that $\pi_1(\mathbb{R}^n, 0)$ is a trivial group (i.e., consists of one element).

32.F. Generalize *32.E* to the situations suggested by *30.H* and *30.4*.

32.5. Calculate the fundamental group of an indiscrete space.

32.6. Calculate the fundamental group of the quotient space of disk D^2 obtained by identifying of each $x \in D^2$ with $-x$.

32.7. Prove that if a two-element space X is path-connected, then X is simply connected.

32.G. Prove that $\pi_1(S^n, (1, 0, \dots, 0))$ with $n \geq 2$ is a trivial group.

Whether you have solved Problem *32.G* or not, we recommend you considering Problems *32.G.1*, *32.G.2*, *32.G.4*, *32.G.5*, and *32.G.6*. They are designed to give an approach to *32.G*, warn about a natural mistake, and prepare an important tool for further calculations of fundamental groups.

32.G.1. Prove that any loop $s : I \to S^n$ that does not fill the entire S^n (i.e., $s(I) \neq S^n$) is null-homotopic, provided that $n \geq 2$. (Cf. Problem *30.7*.)

Warning: for any n, there exists a loop filling S^n. See Problem *10.49x*.

32.G.2. Can a loop filling S^2 be null-homotopic?

32.G.3 Corollary of Lebesgue Lemma 17.W. *Let $s : I \to X$ be a path, Γ an open cover of a topological space X. There exists a sequence of points $a_1, \dots, a_N \in I$ with $0 = a_1 < a_2 < \dots < a_{N-1} < a_N = 1$ such that $s([a_i, a_{i+1}])$ is contained in an element of Γ for each i.*

32.G.4. Prove that if $n \geq 2$, then for any path $s : I \to S^n$ the segment I has a subdivision into a finite number of subintervals such that the restriction of s to each of the subintervals is homotopic to a map with nowhere-dense image via a homotopy fixed on the endpoints of the subinterval.

32.G.5. Prove that if $n \geq 2$, then any loop in S^n is homotopic to a non-surjective loop.

32.G.6. 1) Deduce *32.G* from *32.G.1* and *32.G.5.* 2) Find all points of the proof of *32.G* obtained in this way, where the condition $n \geq 2$ is used.

⌈32′5⌋ Fundamental Group of a Product

32.H. *The fundamental group of the product of topological spaces is canonically isomorphic to the product of the fundamental groups of the factors:*

$$\pi_1(X \times Y, (x_0, y_0)) = \pi_1(X, x_0) \times \pi_1(Y, y_0).$$

32.8. Consider a loop $u : I \to X$ at x_0, a loop $v : I \to Y$ at y_0, and the loop $w = u \times v : I \to X \times Y$. We introduce the loops $u' : I \to X \times Y : t \mapsto (u(t), y_0))$ and $v' : I \to X \times Y : t \mapsto (x_0, v(t))$. Prove that $u'v' \sim w \sim v'u'$.

32.9. Prove that $\pi_1(\mathbb{R}^n \smallsetminus 0, (1, 0, \dots, 0))$ is trivial if $n \geq 3$.

⌈32′6⌋ Simply-Connectedness

A nonempty topological space X is *simply connected* (or *one-connected*) if X is path-connected and every loop in X is null-homotopic.

32.I. *For a path-connected topological space X, the following statements are equivalent:*

(1) *X is simply connected,*

(2) *each continuous map $f : S^1 \to X$ is (freely) null-homotopic,*

(3) *each continuous map $f : S^1 \to X$ extends to a continuous map $D^2 \to X$,*

(4) *any two paths $s_1, s_2 : I \to X$ connecting the same points x_0 and x_1 are homotopic.*

Theorem *32.I* is closely related to Theorem *32.J* below. Notice that since Theorem *32.J* concerns not all loops, but an individual loop, it is applicable in a broader range of situations.

32.J. *Let X be a topological space, $s : S^1 \to X$ a circular loop. Then the following statements are equivalent:*

(1) *s is null-homotopic,*

(2) *s is* freely *null-homotopic,*

(3) *s extends to a continuous map $D^2 \to X$,*

(4) *the paths* $s_+, s_- : I \to X$ *defined by the formula* $s_\pm(t) = s(e^{\pm\pi it})$ *are homotopic.*

32.J.1. Riddle. To prove that 4 statements are equivalent, we must prove at least 4 implications. What implications would you choose for the easiest proof of Theorem *32.J*?

32.J.2. Does homotopy of circular loops imply that these circular loops are free homotopic?

32.J.3. A homotopy between a map of the circle and a constant map possesses a quotient map whose source space is homeomorphic to the disk D^2.

32.J.4. Represent the problem of constructing a homotopy between the paths s_+ and s_- as a problem of extending a certain continuous map of the boundary of a square to the whole square.

32.J.5. When we solve the extension problem obtained as a result of Problem *32.J.4*, does it help to know that the circular loop $S^1 \to X : t \mapsto s(e^{2\pi it})$ extends to a continuous map of a disk?

32.10. Which of the following spaces are simply connected:
(a) a discrete space; (b) an indiscrete space; (c) $\mathbb{R}^n$; (d) a convex set; (e) a star-shaped set; (f) S^n; (g) $\mathbb{R}^n \smallsetminus 0$?

32.11. Prove that if a topological space X is the union of two open simply connected sets U and V with path-connected intersection $U \cap V$, then X is simply connected.

32.12. Show that the assumption in *32.11* that U and V are open is necessary.

32.13*. Let X be a topological space, $U, V \subset X$ two open subsets. Prove that if $U \cup V$ and $U \cap V$ are simply connected, then so are U and V.

⌈32′7x⌋ Fundamental Group of a Topological Group

Let G be a topological group. Given loops $u, v : I \to G$ starting at the unity $1 \in G$, we define a loop $u \odot v : I \to G$ by the formula $u \odot v(t) = u(t) \cdot v(t)$, where $\cdot$ denotes the group operation in G.

32.Kx. Prove that the set $\Omega(G, 1)$ of all loops in G starting at 1 equipped with the operation $\odot$ is a group.

32.Lx. Prove that the operation $\odot$ on $\Omega(G, 1)$ determines a group operation on $\pi_1(G, 1)$, which coincides with the standard group operation (determined by multiplication of paths).

32.Lx.1. For loops $u, v \to G$ starting at 1, find $(ue_1) \odot (e_1v)$.

32.Mx. The fundamental group of a topological group is Abelian.

⌈32′8x⌋ High Homotopy Groups

Let X be a topological space, $x_0 \in X$. A continuous map $I^r \to X$ mapping the boundary ∂I^r of I^r to x_0 is a ***spheroid of dimension*** r of X at x_0, or just an r-***spheroid***. Two r-spheroids are ***homotopic*** if they are ∂I^r-homotopic. For two r-spheroids u and v of X at x_0, $r \geq 1$, define the product uv by the formula

$$uv(t_1, t_2, \dots, t_r) = \begin{cases} u(2t_1, t_2, \dots, t_r) & \text{if } t_1 \in [0, 1/2], \\ v(2t_1 - 1, t_2, \dots, t_r) & \text{if } t_1 \in [1/2, 1]. \end{cases}$$

The set of homotopy classes of r-spheroids of a space X at x_0 is the rth (or r-dimensional) homotopy group $\pi_r(X, x_0)$ of X at x_0. Thus,

$$\pi_r(X, x_0) = \pi(I^r, \partial I^r;\ X, x_0).$$

Multiplication of spheroids induces multiplication in $\pi_r(X, x_0)$, which makes $\pi_r(X, x_0)$ a group.

***32.N*x.** Find $\pi_r(\mathbb{R}^n, 0)$.

***32.O*x.** *For any X and x_0, the group $\pi_r(X, x_0)$ with $r \geq 2$ is Abelian.*

Similar to 32′3, higher-dimensional homotopy groups can be built up not out of homotopy classes of maps $(I^r, \partial I^r) \to (X, x_0)$, but as

$$\pi(S^r, (1, 0, \dots, 0);\ X, x_0).$$

Another way, also quite popular, is to define $\pi_r(X, x_0)$ as

$$\pi(D^r, \partial D^r;\ X, x_0).$$

***32.P*x.** Construct natural bijections

$$\pi(I^r, \partial I^r;\ X, x_0) \to \pi(D^r, \partial D^r;\ X, x_0) \to \pi(S^r, (1, 0, \dots, 0);\ X, x_0).$$

32.Q*x. *Riddle. For any X, x_0 and $r \geq 2$, present group $\pi_r(X, x_0)$ as the fundamental group of some space.

***32.R*x.** Prove the following generalization of *32.H*:

$$\pi_r(X \times Y, (x_0, y_0)) = \pi_r(X, x_0) \times \pi_r(Y, y_0).$$

***32.S*x.** Formulate and prove analogs of Problems *32.Kx* and *32.Lx* for higher homotopy groups and $\pi_0(G, 1)$.

33. The Role of Base Point

⌈33′1⌋ Overview of the Role of Base Point

Sometimes the choice of the base point does not matter, sometimes it is obviously crucial, and sometimes this is a delicate question. In this section, we have to clarify all subtleties related to the base point. We start with preliminary formulations describing the subject in its entirety, but without some necessary details.

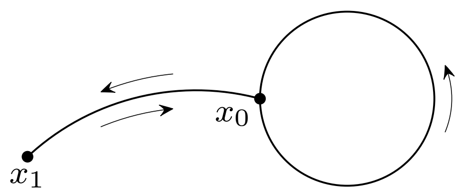
The role of the base point may be roughly described as follows:

- When the base point changes within the same path-connected component, the fundamental group remains in the same class of isomorphic groups.
- However, if the group is non-Abelian, it is impossible to find a natural isomorphism between the fundamental groups at different base points even in the same path-connected component.
- Fundamental groups of a space at base points belonging to different path-connected components have nothing to do with each other.

In this section, these will be demonstrated. The proof involves useful constructions, whose importance extends far outside the frameworks of our initial question on the role of the base point.

⌈33′2⌋ Definition of Translation Maps

Let x_0 and x_1 be two points of a topological space X, and let s be a path connecting x_0 with x_1. Denote by σ the homotopy class $[s]$ of s. Define a map $T_s : \pi_1(X, x_0) \to \pi_1(X, x_1)$ by the formula $T_s(\alpha) = \sigma^{-1}\alpha\sigma$.



33.1. Prove that for any loop $a : I \to X$ representing $\alpha \in \pi_1(X, x_0)$ and any path $s : I \to X$ with $s(0) = x_0$ the loop a is connected with a loop representing $T_s(\alpha)$ by a free homotopy $H : I \times I \to X$ such that $H(0, t) = H(1, t) = s(t)$ for $t \in I$.

33.2. Let $a, b : I \to X$ be two loops homotopic via a homotopy $H : I \times I \to X$ such that $H(0, t) = H(1, t)$ (i.e., H is a free homotopy of loops: at each moment $t \in I$, it keeps the endpoints of the path coinciding). Set $s(t) = H(0, t)$ (hence, s is the path run through by the initial point of the loop under the homotopy). Prove that the homotopy class of b is the image of the homotopy class of a under $T_s : \pi_1(X, s(0)) \to \pi_1(X, s(1))$.

⌈33′3⌋ Properties of T_s

33.A. *T_s is a (group) homomorphism.*[4]

33.B. *If u is a path connecting x_0 to x_1 and v is a path connecting x_1 with x_2, then $T_{uv} = T_v \circ T_u$. In other words, the diagram*

$$\begin{tikzcd} \pi_1(X, x_0) \arrow[r, "T_u"] \arrow[dr, "T_{uv}"'] & \pi_1(X, x_1) \arrow[d, "T_v"] \\ & \pi_1(X, x_2) \end{tikzcd}$$

is commutative.

33.C. *If paths u and v are homotopic, then $T_u = T_v$.*

33.D. $T_{e_a} = \mathrm{id} : \pi_1(X, a) \to \pi_1(X, a)$.

33.E. $T_{s^{-1}} = T_s^{-1}$.

33.F. *T_s is an isomorphism for any path s.*

33.G. *For any points x_0 and x_1 lying in the same path-connected component of X, the groups $\pi_1(X, x_0)$ and $\pi_1(X, x_1)$ are isomorphic.*

Despite the result of Theorem *33.G*, we cannot write $\pi_1(X)$ even if the topological space X is path-connected. The reason is that although the groups $\pi_1(X, x_0)$ and $\pi_1(X, x_1)$ are isomorphic, there may be no canonical isomorphism between them (see *33.J* below).

33.H. *The space X is simply connected iff X is path-connected and the group $\pi_1(X, x_0)$ is trivial for a certain point $x_0 \in X$.*

⌈33′4⌋ Role of Path

33.I. If a loop s represents an element σ of the fundamental group $\pi_1(X, x_0)$, then T_s is the inner automorphism of $\pi_1(X, x_0)$ defined by $\alpha \mapsto \sigma^{-1}\alpha\sigma$.

33.J. *Let x_0 and x_1 be points of a topological space X belonging to the same path-connected component. The isomorphisms $T_s : \pi_1(X, x_0) \to \pi_1(X, x_1)$ do not depend on s iff $\pi_1(X, x_0)$ is an Abelian group.*

Theorem *33.J* implies that if the fundamental group of a topological space X is Abelian, then we may simply write $\pi_1(X)$.

[4] Recall that this means that $T_s(\alpha\beta) = T_s(\alpha)T_s(\beta)$.

⌈33′5x⌋ In Topological Group

In a topological group G, there is another way to relate $\pi_1(G, x_0)$ with $\pi_1(G, x_1)$: there are homeomorphisms $L_g : G \to G : x \mapsto xg$ and $R_g : G \to G : x \mapsto gx$, so that there are two induced isomorphisms $(L_{x_0^{-1}x_1})_* : \pi_1(G, x_0) \to \pi_1(G, x_1)$ and $(R_{x_1x_0^{-1}})_* : \pi_1(G, x_0) \to \pi_1(G, x_1)$.

***33.K*x.** Let G be a topological group, $s : I \to G$ a path. Prove that

$$T_s = (L_{s(0)^{-1}s(1)})_* = (R_{s(1)s(0)^{-1}}) : \pi_1(G, s(0)) \to \pi_1(G, s(1)).$$

***33.L*x.** Deduce from *33.K*x that the fundamental group of a topological group is Abelian (cf. *32.M*x).

***33.3*x.** Prove that the following spaces have Abelian fundamental groups:

(1) the space of nondegenerate real $n \times n$ matrices $GL(n, \mathbb{R}) = \{A \mid \det A \neq 0\}$;
(2) the space of orthogonal real $n \times n$ matrices $O(n, \mathbb{R}) = \{A \mid A \cdot ({}^tA) = \mathbb{E}\}$;
(3) the space of special unitary complex $n \times n$ matrices $SU(n) = \{A \mid A \cdot ({}^t\bar{A}) = 1, \det A = 1\}$.

⌈33′6x⌋ In High Homotopy Groups

33.M*x. *Riddle. Guess how T_s is generalized to $\pi_r(X, x_0)$ with any r.

Here is another form of the same question. We include it because its statement contains a greater piece of an answer.

33.N*x. *Riddle. Given a path $s : I \to X$ with $s(0) = x_0$ and a spheroid $f : I^r \to X$ at x_0, how does one make up a spheroid at $x_1 = s(1)$ out of these?

***33.O*x.** Let $s : I \to X$ be a path, $f : I^r \to X$ a spheroid with $f(\operatorname{Fr} I^r) = s(0)$. Prove that there exists a homotopy $H : I^r \times I \to X$ of f such that $H(\operatorname{Fr} I^r \times t) = s(t)$ for any $t \in I$. Furthermore, the spheroid obtained by such a homotopy is unique up to homotopy and determines an element of $\pi_r(X, s(1))$, which is uniquely determined by the homotopy class of s and the element of $\pi_r(X, s(0))$ represented by f.

Certainly, a solution of *33.O*x gives an answer to *33.N*x and *33.M*x. The map $\pi_r(X, s(0)) \to \pi_r(X, s(1))$ defined by *33.O*x is denoted by T_s. By *33.2*, this T_s generalizes T_s defined in the beginning of the section for the case $r = 1$.

***33.P*x.** Prove that the properties of T_s formulated in Problems *33.A*–*33.F* hold true in all dimensions.

33.Q*x. *Riddle. What are the counterparts of *33.K*x and *33.L*x for higher homotopy groups?

Chapter VII

Covering Spaces and Calculation of Fundamental Groups

34. Covering Spaces

⌈34′1⌋ Definition of Covering

Let X and B be topological spaces, $p : X \to B$ a continuous map. Assume that p is surjective and each point of B possesses a neighborhood U such that the preimage $p^{-1}(U)$ of U is a disjoint union of open sets V_α and p homeomorphically maps each V_α onto U. Then $p : X \to B$ is a *covering* (of B), the space B is the *base* of this covering, X is the *covering space* for B and the *total space* of the covering. Neighborhoods like U are said to be *trivially covered*. The map p is a *covering map* or *covering projection*.

34.A. Let B be a topological space, F a discrete space. Prove that the projection $\mathrm{pr}_B : B \times F \to B$ is a covering.

> ***34.1.*** If $U' \subset U \subset B$ and the neighborhood U is trivially covered, then the neighborhood U' is also trivially covered.

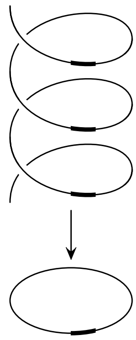
The following statement shows that in a certain sense any covering locally is organized as the covering of *34.A*.

34.B. A continuous surjective map $p : X \to B$ is a covering iff for each point a of B the preimage $p^{-1}(a)$ is discrete and there exist a neighborhood U of a

and a homeomorphism $h : p^{-1}(U) \to U \times p^{-1}(a)$ such that $p|_{p^{-1}(U)} = \mathrm{pr}_U \circ h$. Here, as usual, $\mathrm{pr}_U : U \times p^{-1}(a) \to U$.

However, the coverings of *34.A* are not interesting. They are *trivial*. Here is the first really interesting example.

34.C. Prove that the map $\mathbb{R} \to S^1 : x \mapsto e^{2\pi i x}$ is a covering.



To distinguish the most interesting examples, a covering with a connected total space is called a covering in the *narrow sense*. Of course, the covering of *34.C* is a covering in the narrow sense.

⌈34′2⌋ More Examples

34.D. The map $\mathbb{R}^2 \to S^1 \times \mathbb{R} : (x, y) \mapsto (e^{2\pi i x}, y)$ is a covering.

34.E. Prove that if $p : X \to B$ and $p' : X' \to B'$ are coverings, then so is $p \times p' : X \times X' \to B \times B'$.

If $p : X \to B$ and $p' : X' \to B'$ are two coverings, then $p \times p' : X \times X' \to B \times B'$ is the *product of the coverings* p and p'. The first example of the product of coverings is presented in *34.D*.

34.F. The map $\mathbb{C} \to \mathbb{C} \smallsetminus 0 : z \mapsto e^z$ is a covering.

34.2. ***Riddle.*** In what sense are the coverings of *34.D* and *34.F* the same? Define an appropriate equivalence relation for coverings.

34.G. The map $\mathbb{R}^2 \to S^1 \times S^1 : (x, y) \mapsto (e^{2\pi i x}, e^{2\pi i y})$ is a covering.

34.H. For any positive integer n, the map $S^1 \to S^1 : z \mapsto z^n$ is a covering.

34.3. Prove that for each positive integer n the map $\mathbb{C} \smallsetminus 0 \to \mathbb{C} \smallsetminus 0 : z \mapsto z^n$ is a covering.

34.I. For any positive integers p and q, the map $S^1 \times S^1 \to S^1 \times S^1 : (z, w) \mapsto (z^p, w^q)$ is a covering.

34.J. The natural projection $S^n \to \mathbb{R}P^n$ is a covering.

34.K. Is $(0,3) \to S^1 : x \mapsto e^{2\pi i x}$ a covering? (Cf. *34.14*.)

34.L. Is the projection $\mathbb{R}^2 \to \mathbb{R} : (x,y) \mapsto x$ a covering? Indeed, why isn't an open interval $(a,b) \subset \mathbb{R}$ a trivially covered neighborhood: its preimage $(a,b) \times \mathbb{R}$ is the union of open intervals $(a,b) \times \{y\}$, which are homeomorphically projected onto (a,b) by the projection $(x,y) \mapsto x$?

34.4. Find coverings of the Möbius strip by a cylinder.

34.5. Find nontrivial coverings of the Möbius strip by itself.

34.6. Find a covering of the Klein bottle by a torus. Cf. Problem *22.14*.

34.7. Find coverings of the Klein bottle by the plane $\mathbb{R}^2$ and the cylinder $S^1 \times \mathbb{R}$, and a nontrivial covering of the Klein bottle by itself.

34.8. Describe explicitly the partition of $\mathbb{R}^2$ into preimages of points under this covering.

34.9*. Find a covering of a sphere with any number of cross-caps by a sphere with handles.

⌈34′3⌋ Local Homeomorphisms versus Coverings

34.10. Any covering is an open map.[1]

A map $f : X \to Y$ is a *local homeomorphism* if each point of X has a neighborhood U such that the image $f(U)$ is open in Y and the submap $\mathrm{ab}(f) : U \to f(U)$ is a homeomorphism.

34.11. Any covering is a local homeomorphism.

34.12. Find a local homeomorphism which is not a covering.

34.13. Prove that the restriction of a local homeomorphism to an open set is a local homeomorphism.

34.14. For which subsets of $\mathbb{R}$ is the restriction of the map of Problem *34.C* a covering?

34.15. Find a nontrivial covering $X \to B$ with X homeomorphic to B and prove that it satisfies the definition of a covering.

⌈34′4⌋ Number of Sheets

Let $p : X \to B$ be a covering. The cardinality (i.e., the number of points) of the preimage $p^{-1}(a)$ of a point $a \in B$ is the *multiplicity* of the covering at a or the *number of sheets of the covering over* a.

34.M. *If the base of a covering is connected, then the multiplicity of the covering at a point does not depend on the point.*

[1] We remind the reader that a map is *open* if the image of any open set is open.

In the case of a covering with connected base, the multiplicity is called the ***number of sheets*** of the covering. If the number of sheets is n, then the covering is n-***sheeted***, and we speak about an n-***fold*** covering. Of course, if the covering is nontrivial, it is impossible to distinguish the sheets of it, but this does not prevent us from speaking about the number of sheets. On the other hand, we adopt the following agreement. By definition, the preimage $p^{-1}(U)$ of any trivially covered neighborhood $U \subset B$ splits into open subsets: $p^{-1}(U) = \cup V_\alpha$, such that the restriction $p|_{V_\alpha} : V_\alpha \to U$ is a homeomorphism. Each of the subsets V_α is a ***sheet*** over U.

34.16. What are the numbers of sheets for the coverings from Section 34′2?

In Problems *34.17–34.19*, we did not assume that you would rigorously justify your answers. This is done below, see Problems *40.3–40.6*.

34.17. What numbers can you realize as the number of sheets of a covering of the Möbius strip by the cylinder $S^1 \times I$?

34.18. What numbers can you realize as the number of sheets of a covering of the Möbius strip by itself?

34.19. What numbers can you realize as the number of sheets of a covering of the Klein bottle by a torus?

34.20. What numbers can you realize as the number of sheets of a covering of the Klein bottle by itself?

34.21. Construct a d-fold covering of a sphere with p handles by a sphere with $1 + d(p - 1)$ handles.

34.22. Let $p : X \to Y$ and $q : Y \to Z$ be coverings. Prove that if q has finitely many sheets, then $q \circ p : x \to Y$ is a covering.

34.23*. Is the hypothesis of finiteness of the number of sheets in Problem *34.22* necessary?

34.24. Let $p : X \to B$ be a covering with compact base B. 1) Prove that if X is compact, then the covering is finite-sheeted. 2) If B is Hausdorff and the covering is finite-sheeted, then X is compact.

34.25. Let X be a topological space presentable as the union of two open connected sets U and V. Prove that if the intersection $U \cap V$ is disconnected, then X has a connected infinite-sheeted covering.

⌈34′5⌋ Universal Coverings

A covering $p : X \to B$ is ***universal*** if X is simply connected. The appearance of the word *universal* in this context is explained below in Section 40.

34.N. Which coverings of the problems stated above in this section are universal?

35. Theorems on Path Lifting

⌈35′1⌋ Lifting

Let $p : X \to B$ and $f : A \to B$ be arbitrary maps. A map $g : A \to X$ such that $p \circ g = f$ is said to *cover* f or be a *lift* of f. Various topological problems can be phrased in terms of finding a continuous lift of some continuous map. Problems of this sort are called *lifting problems*. They may involve additional requirements. For example, the required lift must coincide with a lift already given on some subspace.

35.A. The identity map $S^1 \to S^1$ does not admit a continuous lifting with respect to the covering $\mathbb{R} \to S^1 : x \mapsto e^{2\pi i x}$. (In other words, there is no continuous map $g : S^1 \to \mathbb{R}$ such that $e^{2\pi i g(x)} = x$ for $x \in S^1$.)

⌈35′2⌋ Path Lifting

35.B Path Lifting Theorem. *Let $p : X \to B$ be a covering, and let $x_0 \in X$ and $b_0 \in B$ be points such that $p(x_0) = b_0$. Then for any path $s : I \to B$ starting at b_0 there is a unique path $\tilde{s} : I \to X$ that starts at x_0 and is a lift of s. (In other words, there exists a unique path $\tilde{s} : I \to X$ with $\tilde{s}(0) = x_0$ and $p \circ \tilde{s} = s$.)*

We can also prove a more general assertion than Theorem *35.B*: see Problems *35.1–35.3*.

35.1. Let $p : X \to B$ be a trivial covering. Then any continuous map f of any space A to B has a continuous lift $\tilde{f} : A \to X$.

35.2. Let $p : X \to B$ be a trivial covering, and let $x_0 \in X$ and $b_0 \in B$ be two points such that $p(x_0) = b_0$. Then any continuous map $f : A \to B$ sending a point a_0 to b_0 has a unique continuous lift $\tilde{f} : A \to X$ with $\tilde{f}(a_0) = x_0$.

35.3. Let $p : X \to B$ be a covering, and let A be a connected and locally connected space. If $f, g : A \to X$ are two continuous maps coinciding at some point and $p \circ f = p \circ g$, then $f = g$.

35.4. If we replace x_0, b_0, and a_0 in Problem *35.2* by pairs of points, then the lifting problem may happen to have no solution $\tilde{f}$ with $\tilde{f}(a_0) = x_0$. Formulate a condition necessary and sufficient for existence of such a solution.

35.5. What goes wrong with the Path Lifting Theorem *35.B* for the local homeomorphism of Problem *34.K*?

35.6. Consider the covering $\mathbb{C} \to \mathbb{C} \smallsetminus 0 : z \mapsto e^z$. Find lifts of the paths $u(t) = 2 - t$ and $v(t) = (1 + t)e^{2\pi i t}$ and their products uv and vu.

⌈35′3⌋ Homotopy Lifting

35.C Path Homotopy Lifting Theorem. *Let $p : X \to B$ be a covering, and let $x_0 \in X$ and $b_0 \in B$ be points such that $p(x_0) = b_0$. Let $u, v : I \to B$*

be paths starting at b_0, *and let* $\tilde{u}, \tilde{v} : I \to X$ *be the lifting paths for* u *and* v *starting at* x_0. *If the paths* u *and* v *are homotopic, then the covering paths* $\tilde{u}$ *and* $\tilde{v}$ *are homotopic.*

35.D Important Corollary. *Under the assumptions of Theorem 35.C, the covering paths* $\tilde{u}$ *and* $\tilde{v}$ *have the same final point (i.e.,* $\tilde{u}(1) = \tilde{v}(1)$*).*

Notice that the paths in *35.C* and *35.D* are assumed to share the initial point x_0. In the statement of *35.D*, we emphasize that they also share the final point.

35.E Corollary of 35.D. *Let* $p : X \to B$ *be a covering,* $s : I \to B$ *a loop. If* s *has a lift* $\tilde{s} : I \to X$ *with* $\tilde{s}(0) \neq \tilde{s}(1)$ *(i.e., there exists a covering path which is not a loop), then* s *is not null-homotopic.*

35.F. If a path-connected space B has a nontrivial path-connected covering space, then the fundamental group of B is nontrivial.

35.7. Prove that any covering $p : X \to B$ with simply connected B and path-connected X is a homeomorphism.

35.8. What corollaries can you deduce from *35.F* and the examples of coverings presented above in Section 34?

35.9. Riddle. Is it really important in the hypothesis of Theorem *35.C* that u and v are paths? To what class of maps can you generalize this theorem?

36. Calculation of Fundamental Groups by Using Universal Coverings

⌈36′1⌋ Fundamental Group of Circle

For an integer n, denote by s_n the loop in S^1 defined by the formula $s_n(t) = e^{2\pi i n t}$. The initial point of this loop is 1. Denote the homotopy class of s_1 by α. Thus, $\alpha \in \pi_1(S^1, 1)$.

36.A. The loop s_n represents $\alpha^n \in \pi_1(S^1, 1)$.

36.B. Find the paths in $\mathbb{R}$ starting at $0 \in \mathbb{R}$ and covering the loops s_n with respect to the universal covering $\mathbb{R} \to S^1$.

36.C. *The homomorphism $\mathbb{Z} \to \pi_1(S^1, 1) : \; n \mapsto \alpha^n$ is an isomorphism.*

36.C.1. The formula $n \mapsto \alpha^n$ determines a homomorphism $\mathbb{Z} \to \pi_1(S^1, 1)$.

36.C.2. Prove that a loop $s : I \to S^1$ starting at 1 is homotopic to s_n if the path $\tilde{s} : I \to \mathbb{R}$ covering s and starting at $0 \in \mathbb{R}$ ends at $n \in \mathbb{R}$ (i.e., $\tilde{s}(1) = n$).

36.C.3. Prove that if the loop s_n is null-homotopic, then $n = 0$.

36.1. Find the image of the homotopy class of the loop $t \mapsto e^{2\pi i t^2}$ under the isomorphism of Theorem *36.C*.

Denote by deg the isomorphism inverse to the isomorphism of Theorem *36.C*.

36.2. For any loop $s : I \to S^1$ starting at $1 \in S^1$, the integer $\deg([s])$ is the final point of the path starting at $0 \in \mathbb{R}$ and covering s.

36.D Corollary of Theorem 36.C. *The fundamental group of $(S^1)^n$ is a free Abelian group of rank n (i.e., isomorphic to $\mathbb{Z}^n$).*

36.E. On the torus $S^1 \times S^1$, find two loops whose homotopy classes generate the fundamental group of the torus.

36.F Corollary of Theorem 36.C. *The fundamental group of the punctured plane $\mathbb{R}^2 \smallsetminus 0$ is an infinite cyclic group.*

36.3. Solve Problems *36.D–36.F* without reference to Theorems *36.C* and *32.H*, but using explicit constructions of the corresponding universal coverings.

⌈36′2⌋ Fundamental Group of Projective Space

The fundamental group of the projective line is an infinite cyclic group. It is calculated in the previous subsection since the projective line is a circle. The zero-dimensional projective space is a point, and hence its fundamental group is trivial. Now we calculate the fundamental groups of projective spaces of all other dimensions.

Let $n \geq 2$, and let $l : I \to \mathbb{R}P^n$ be a loop covered by a path $\tilde{l} : I \to S^n$ which connects two antipodal points of S^n, say, the poles $P_+ = (1, 0, \dots, 0)$ and $P_- = (-1, 0, \dots, 0)$. Denote by λ the homotopy class of l. It is an element of $\pi_1(\mathbb{R}P^n, (1 : 0 : \dots : 0))$.

36.G. *For any $n \geq 2$, the group $\pi_1(\mathbb{R}P^n, (1 : 0 : \dots : 0))$ is a cyclic group of order 2. It has two elements: λ and 1.*

36.G.1 Lemma. *Any loop in $\mathbb{R}P^n$ at $(1 : 0 : \dots : 0)$ is homotopic either to l or constant. This depends on whether the covering path of the loop connects the poles P_+ and P_-, or is a loop.*

36.4. Where did we use the assumption $n \geq 2$ in the proofs of Theorem *36.G* and Lemma *36.G.1*?

⌈36′3⌋ Fundamental Group of Bouquet of Circles

Consider a family of topological spaces $\{X_\alpha\}$. In each of the spaces, we mark a point x_α. Take the disjoint sum $\bigsqcup_\alpha X_\alpha$ and identify all marked points. The resulting quotient space $\bigvee_\alpha X_\alpha$ is the *bouquet* of $\{X_\alpha\}$. Hence, a *bouquet of q circles* is a space which is the union of q copies of a circle. The copies meet at a single common point, and this is the only common point for any two of them. The common point is the *center* of the bouquet.

Denote the bouquet of q circles by B_q and its center by c. Let $u_1, \dots, u_q$ be loops in B_q starting at c and parameterizing the q copies of the circle that constitute B_q. Denote by α_i the homotopy class of u_i.

36.H. *$\pi_1(B_q, c)$ is a free group freely generated by $\alpha_1, \dots, \alpha_q$.*

⌈36′4⌋ Algebraic Digression: Free Groups

Recall that a group G is a *free group freely generated* by its elements $a_1, \dots, a_q$ if:

- each element $x \in G$ is a product of powers (with positive or negative integer exponents) of $a_1, \dots, a_q$, i.e.,

$$x = a_{i_1}^{e_1} a_{i_2}^{e_2} \dots a_{i_n}^{e_n}$$

  and
- this expression is unique up to the following trivial ambiguity: we can insert or delete factors $a_i a_i^{-1}$ and $a_i^{-1} a_i$ or replace a_i^m by $a_i^r a_i^s$ with $r + s = m$.

36.I. *A free group is determined up to isomorphism by the number of its free generators.*

The number of free generators is the ***rank*** of the free group. For a standard representative of the isomorphism class of free groups of rank q, we can take the group of words in an alphabet of q letters $a_1, \dots, a_q$ and their inverses $a_1^{-1}, \dots, a_q^{-1}$. Two words represent the same element of the group iff they are obtained from each other by a sequence of insertions or deletions of fragments $a_i a_i^{-1}$ and $a_i^{-1} a_i$. This group is denoted by $\mathbb{F}(a_1, \dots, a_q)$, or just $\mathbb{F}_q$ if the notation for the generators is not to be emphasized.

36.J. *Each element of $\mathbb{F}(a_1, \dots, a_q)$ has a unique shortest representative. This is a word without fragments that could have been deleted.*

The number $l(x)$ of letters in the shortest representative of an element $x \in \mathbb{F}(a_1, \dots, a_q)$ is the ***length*** of x. Certainly, this number is not well defined, unless the generators are fixed.

> ***36.5.*** Show that an automorphism of $\mathbb{F}_q$ can map $x \in \mathbb{F}_q$ to an element with different length. For what value of q does such an example not exist? Is it possible to change the length in this way arbitrarily?

36.K. *A group G is a free group freely generated by its elements $a_1, \dots, a_q$ iff every map of the set $\{a_1, \dots, a_q\}$ to any group X extends to a unique homomorphism $G \to X$.*

Theorem *36.K* is sometimes taken as a definition of a free group. (Definitions of this sort emphasize relations among different groups, rather than the internal structure of a single group. Of course, relations among groups can tell everything about the "internal affairs" of each group.)

Now we can reformulate Theorem *36.H* as follows:

36.L. *The homomorphism*

$$\mathbb{F}(a_1, \dots, a_q) \to \pi_1(B_q, c)$$

taking a_i to α_i for $i = 1, \dots, q$ is an isomorphism.

First, for the sake of simplicity we restrict ourselves to the case where $q = 2$. This allows us to avoid superfluous complications in notation and pictures. This is the simplest case that really represents the general situation. The case $q = 1$ is too special.

To take advantages of this, we change the notation and put $B = B_2$, $u = u_1$, $v = u_2$, $\alpha = \alpha_1$, and $\beta = \alpha_2$.

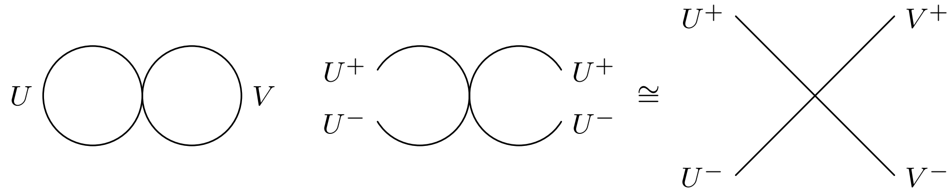
Now Theorem *36.L* looks as follows:

The homomorphism $\mathbb{F}(a, b) \to \pi(B, c)$ taking a to α and b to β is an isomorphism.

This theorem can be proved like Theorems *36.C* and *36.G*, provided that we know the universal covering of B.

⌈36′5⌋ Universal Covering for Bouquet of Circles

Denote by U and V the points antipodal to c on the circles of B. Cut B at these points, removing U and V and replacing each of them with two new points. Whatever this operation is, its result is a cross K, which is the union of four closed segments with a common endpoint c. There appears a natural map $P : K \to B$ that sends the center c of the cross to the center c of B and homeomorphically maps the rays of the cross onto half-circles of B. Since the circles of B are parameterized by loops u and v, the halves of each of the circles are ordered: the corresponding loop passes first one of the halves and then the other one. Denote by U^+ the point of $P^{-1}(U)$ belonging to the ray mapped by P onto the second half of the circle, and by U^- the other point of $P^{-1}(U)$. We similarly denote points of $P^{-1}(V)$ by V^+ and V^-.



The restriction of P to $K \smallsetminus \{U^+, U^-, V^+, V^-\}$ homeomorphically maps this set onto $B \smallsetminus \{U, V\}$. Therefore, P provides a covering of $B \smallsetminus \{U, V\}$. However, it fails to be a covering at U and V: none of these points has a trivially covered neighborhood. Furthermore, the preimage of each of these points consists of 2 points (the endpoints of the cross), where P is not even a local homeomorphism. To eliminate this defect, we attach a copy of K at each of the 4 endpoints of K and extend P in a natural way to the result. But then 12 new endpoints appear at which the map is not a local homeomorphism. Well, we repeat the trick and restore the property of being a local homeomorphism at each of the 12 new endpoints. Then we do this at each of the 36 new points, etc. However, if we repeat this infinitely many times, all bad points become nice ones.[2]

36.M. Formalize the construction of a covering for B described above.

[2] This sounds like a story about a battle with Hydra, but the happy ending demonstrates that modern mathematicians have a magic power of the sort that the heroes of myths and tales could not even dream of. Indeed, we meet a Hydra K with 4 heads, chop off all the heads, but, according to the old tradition of the genre, 3 new heads appear in place of each of the original heads. We chop them off, and the story repeats. We do not even try to prevent this multiplication of heads. We just chop them off. But contrary to the real heroes of tales, we act outside Time and hence have no time limitations. Thus, after infinitely many repetitions of the exercise with an exponentially growing number of heads, we succeed! No heads left!

This is a typical success story about an infinite construction in mathematics. Sometimes, as in our case, such a construction can be replaced by a finite one, which, however, deals with infinite objects. Nevertheless, there are important constructions where an infinite fragment is unavoidable.

Consider $\mathbb{F}(a,b)$ as a discrete topological space. Take $K \times \mathbb{F}(a,b)$. The latter space can be thought of as a collection of copies of K enumerated by elements of $\mathbb{F}(a,b)$. Topologically, this is a disjoint sum of the copies because $\mathbb{F}(a,b)$ is equipped with discrete topology. In $K \times \mathbb{F}(a,b)$, we identify points (U^-, g) with (U^+, ga) and (V^-, g) with (V^+, gb) for each $g \in \mathbb{F}(a,b)$. Denote the resulting quotient space by X.

36.N. The composition of the projection $K \times \mathbb{F}(a,b) \to K$ and $P : K \to B$ determines a continuous quotient map $p : X \to B$.

36.O. $p : X \to B$ is a covering.

36.P. X is path-connected. For any $g \in \mathbb{F}(a,b)$, there is a path connecting $(c, 1)$ with (c, g) and covering the loop obtained from g by replacing a with u and b with v.

36.Q. X is simply connected.

36.R*. Let a topological space X be the union of two open path-connected sets U and V. Prove that if $U \cap V$ has at least three connected components, then the fundamental group of X is non-Abelian and, moreover, admits an epimorphism onto a free group of rank 2.

⌈36′6⌋ Fundamental Groups of Finite Topological Spaces

36.6. Prove that if a three-element space X is path-connected, then X is simply connected (cf. *32.7*).

36.7. Consider a topological space $X = \{a, b, c, d\}$ with topology determined by the base $\{\{a\}, \{c\}, \{a, b, c\}, \{c, d, a\}\}$. Prove that X is path-connected, but not simply connected.

36.8. Calculate $\pi_1(X)$.

36.9. Let X be a finite topological space with nontrivial fundamental group. Let n_0 be the least possible cardinality of X. 1) Find n_0. 2) What nontrivial groups arise as fundamental groups of n_0-element spaces?

36.10. 1) Find a finite topological space with non-Abelian fundamental group. 2) What is the least possible cardinality of such a space?

36.11*. Find a finite topological space with fundamental group isomorphic to $\mathbb{Z}_2$.

Chapter VIII

Fundamental Group and Maps

37. Induced Homomorphisms and Their First Applications

⌈37′1⌋ Homomorphisms Induced by a Continuous Map

Let $f : X \to Y$ be a continuous map of a topological space X to a topological space Y. Let $x_0 \in X$ and $y_0 \in Y$ be points such that $f(x_0) = y_0$. The latter property of f is expressed by saying that f *maps the pair* (X, x_0) *to the pair* (Y, y_0), and writing $f : (X, x_0) \to (Y, y_0)$.

Consider the map $f_\# : \Omega(X, x_0) \to \Omega(Y, y_0) : s \mapsto f \circ s$. This map assigns to a loop its composition with f.

37.A. *The map $f_\#$ sends homotopic loops to homotopic loops.*

Therefore, $f_\#$ induces a map $f_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$.

37.B. *$f_* : \pi(X, x_0) \to \pi_1(Y, y_0)$ is a homomorphism for any continuous map $f : (X, x_0) \to (Y, y_0)$.*

$f_* : \pi(X, x_0) \to \pi_1(Y, y_0)$ is the *homomorphism induced* by f.

37.C. *Let $f : (X, x_0) \to (Y, y_0)$ and $g : (Y, y_0) \to (Z, z_0)$ be (continuous) maps. Then we have*

$$(g \circ f)_* = g_* \circ f_* : \pi_1(X, x_0) \to \pi_1(Z, z_0).$$

37.D. *Let $f, g : (X, x_0) \to (Y, y_0)$ be two continuous maps homotopic via a homotopy fixed at x_0. Then $f_* = g_*$.*

37.E. Riddle. How can we generalize Theorem *37.D* to the case of freely homotopic f and g?

37.F. *Let $f : X \to Y$ be a continuous map, and let x_0 and x_1 be two points of X connected by a path $s : I \to X$. Denote $f(x_0)$ by y_0 and $f(x_1)$ by y_1. Then the diagram*

$$\begin{array}{ccc} \pi_1(X, x_0) & \xrightarrow{f_*} & \pi_1(Y, y_0) \\ {\scriptstyle T_s}\downarrow & & \downarrow{\scriptstyle T_{f\circ s}} \\ \pi_1(X, x_1) & \xrightarrow{f_*} & \pi_1(Y, y_1) \end{array}$$

is commutative, i.e., $T_{f\circ s} \circ f_* = f_* \circ T_s$.

37.1. Prove that the map $\mathbb{C} \setminus 0 \to \mathbb{C} \setminus 0 : z \mapsto z^3$ is not homotopic to the identity map $\mathbb{C} \setminus 0 \to \mathbb{C} \setminus 0 : z \mapsto z$.

37.2. Let X be a subset of $\mathbb{R}^n$. Prove that if a continuous map $f : X \to Y$ extends to a continuous map $\mathbb{R}^n \to Y$, then $f_* : \pi_1(X, x_0) \to \pi_1(Y, f(x_0))$ is a trivial homomorphism (i.e., sends everything to the unit) for each $x_0 \in X$.

37.3. Prove that if a Hausdorff space X contains an open set homeomorphic to $S^1 \times S^1 \setminus (1, 1)$, then X has infinite noncyclic fundamental group.

37.3.1. Prove that a space X satisfying the conditions of *37.3* can be continuously mapped to a space with infinite noncyclic fundamental group in such a way that the map would induce an epimorphism of $\pi_1(X)$ onto this infinite group.

37.4. Prove that the space $GL(n, \mathbb{C})$ of complex $n \times n$ matrices with nonzero determinant has infinite fundamental group.

⌈37′2⌋ Fundamental Theorem of Algebra

Our goal here is to prove the following theorem, which, at first glance, has no relation to fundamental group.

37.G Fundamental Theorem of Algebra. *Every polynomial of positive degree in one variable with complex coefficients has a complex root.*

In more detail:

Let $p(z) = z^n + a_1 z^{n-1} + \dots + a_n$ be a polynomial of degree $n > 0$ in z with complex coefficients. Then there exists a complex number w such that $p(w) = 0$.

Although it is formulated in an algebraic way and called "The Fundamental Theorem of Algebra," it has no simple algebraic proof. Its proofs usually involve topological arguments or use complex analysis. This is so because the field $\mathbb{C}$ of complex numbers as well as the field $\mathbb{R}$ of reals is extremely difficult to describe in purely algebraic terms: all customary constructive descriptions involve a sort of completion construction, cf. Section 18.

37.G.1 Reduction to Problem on a Map. Deduce Theorem *37.G* from the following statement:

For any complex polynomial $p(z)$ of a positive degree, the image of the map $\mathbb{C} \to \mathbb{C} : z \mapsto p(z)$ contains the zero. In other words, the formula $z \mapsto p(z)$ does not determine a map $\mathbb{C} \to \mathbb{C} \smallsetminus 0$.

37.G.2 Estimate of Remainder. Let $p(z) = z^n + a_1 z^{n-1} + \dots + a_n$ be a complex polynomial, $q(z) = z^n$, and $r(z) = p(z) - q(z)$. Then there exists a positive real R such that $|r(z)| < |q(z)| = R^n$ for any z with $|z| = R$.

37.G.3 Lemma on Lady with Dog. (Cf. *30.11*.) A lady $q(z)$ and her dog $p(z)$ walk on the punctured plane $\mathbb{C} \smallsetminus 0$ periodically (i.e., say, with $z \in S^1$). Prove that if the lady does not let the dog run further than $|q(z)|$ from her, then the doggy's loop $S^1 \to \mathbb{C} \smallsetminus 0 : z \mapsto p(z)$ is homotopic to the lady's loop $S^1 \to \mathbb{C} \smallsetminus 0 : z \mapsto q(z)$.

37.G.4 Lemma for Dummies. (Cf. *30.12*.) If $f : X \to Y$ is a continuous map and $s : S^1 \to X$ is a null-homotopic loop, then $f \circ s : S^1 \to Y$ is also null-homotopic.

⌈37′3x⌋ Generalization of Intermediate Value Theorem

37.Hx. Riddle. How to generalize Intermediate Value Theorem *13.A* to the case of maps $f : D^n \to \mathbb{R}^n$?

37.Ix. Find out whether Intermediate Value Theorem *13.A* is equivalent to the following statement:
Let $f : D^1 \to \mathbb{R}^1$ be a continuous map. If $0 \notin f(S^0)$ and the submap $f|_{S^0} : S^0 \to \mathbb{R}^1 \smallsetminus 0$ of f induces a nonconstant map $\pi_0(S^0) \to \pi_0(\mathbb{R}^1 \smallsetminus 0)$, then there exists a point $x \in D^1$ such that $f(x) = 0$.

37.Jx. Riddle. Suggest a generalization of Intermediate Value Theorem to maps $D^n \to \mathbb{R}^n$ which would generalize its reformulation *37.Ix*. To do it, you must define the induced homomorphism for homotopy groups.

37.Kx. *Let $f : D^n \to \mathbb{R}^n$ be a continuous map. If $f(S^{n-1})$ does not contain $0 \in \mathbb{R}^n$ and the submap $f|_{S^{n-1}} : S^{n-1} \to \mathbb{R}^n \smallsetminus 0$ of f induces a nonconstant map*

$$\pi_{n-1}(S^{n-1}) \to \pi_{n-1}(\mathbb{R}^n \smallsetminus 0),$$

then there exists a point $x \in D^1$ such that $f(x) = 0$.

Usability of Theorem *37.Kx* is impeded by a condition which is difficult to check if $n > 0$. For $n = 1$, this is still possible in the framework of the theory developed above.

***37.5*x.** Let $f : D^2 \to \mathbb{R}^2$ be a continuous map. If $f(S^1)$ does not contain $a \in \mathbb{R}^2$ and the circular loop $f|_{S^1} : S^1 \to \mathbb{R}^2 \smallsetminus a$ determines a nontrivial element of $\pi_1(\mathbb{R}^2 \smallsetminus a)$, then there exists $x \in D^2$ such that $f(x) = a$.

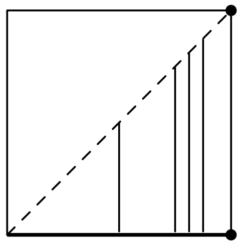
***37.6*x.** Let $f : D^2 \to \mathbb{R}^2$ be a continuous map that leaves fixed each point of the boundary circle S^1. Then $f(D^2) \supset D^2$.

***37.7*x.** Assume that $f : \mathbb{R}^2 \to \mathbb{R}^2$ is a continuous map and there exists a real number m such that $|f(x) - x| \le m$ for any $x \in \mathbb{R}^2$. Prove that f is a surjection.

***37.8*x.** Let $u, v : I \to I \times I$ be two paths such that $u(0) = (0,0)$, $u(1) = (1,1)$, $v(0) = (0,1)$, and $v(1) = (1,0)$. Prove that $u(I) \cap v(I) \neq \varnothing$.

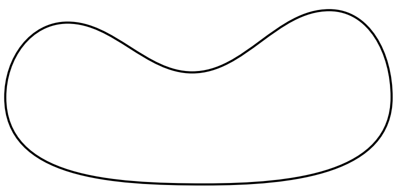
37.8*x*.1. Let u and v be as in *37.8*x. Prove that $0 \in \mathbb{R}^2$ is a value of the map $w : I^2 \to \mathbb{R}^2 : (x,y) \mapsto u(x) - v(y)$.

***37.9*x.** Prove that there exist disjoint connected sets $F, G \subset I^2$ such that the corner points $(0,0)$ and $(1,1)$ of the square I^2 belong to F, while $(0,1)$, $(1,0) \in G$.



***37.10*x.** In addition, can we require that the sets F and G satisfying the assumptions of Problem *37.9*x be closed?

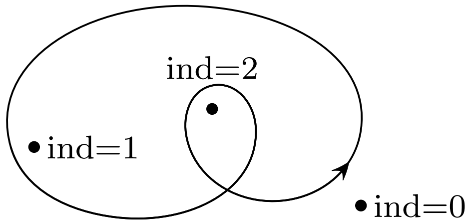
***37.11*x.** Let C be a smooth simple closed curve on the plane with two inflection points having the form shown in the figure. Prove that there is a line intersecting C at four points a, b, c, and d with segments $[a,b]$, $[b,c]$, and $[c,d]$ of the same length.



⌈37′4x⌋ Winding Number

As we know (see *36.F*), the fundamental group of the punctured plane $\mathbb{R}^2 \smallsetminus 0$ is isomorphic to $\mathbb{Z}$. There are two isomorphisms, which differ by multiplication by -1. Choose one of them that sends the homotopy class of the loop $t \mapsto (\cos 2\pi t, \sin 2\pi t)$ to $1 \in \mathbb{Z}$. In terms of circular loops, the isomorphism means that each loop $f : S^1 \to \mathbb{R}^2 \smallsetminus 0$ is assigned an integer. Roughly speaking, it is the number of times the loop goes around 0 (with account of direction).

Now we change the viewpoint in this consideration: we fix the loop, but vary the point. Let $f : S^1 \to \mathbb{R}^2$ be a circular loop and let $x \in \mathbb{R}^2 \smallsetminus f(S^1)$. Then f determines an element in $\pi_1(\mathbb{R}^2 \smallsetminus x) = \mathbb{Z}$. (Here we choose basically the same identification of $\pi_1(\mathbb{R}^2 \smallsetminus x)$ with $\mathbb{Z}$ that sends 1 to the homotopy class of $t \mapsto x + (\cos 2\pi t, \sin 2\pi t)$.) This number is denoted by $\operatorname{ind}(f, x)$ and called the *winding number* or *index* of x with respect to f.



It is also convenient to characterize the number $\operatorname{ind}(u, x)$ as follows. Along with the circular loop $u : S^1 \to \mathbb{R}^2 \smallsetminus x$, consider the map $\varphi_{u,x} : S^1 \to S^1 : z \mapsto (u(z) - x)/|u(z) - x|$. The homomorphism $(\varphi_{u,x})_* : \pi_1(S^1) \to \pi_1(S^1)$ sends the generator α of the fundamental group of the circle to the element $k\alpha$, where $k = \operatorname{ind}(u, x)$.

37.Lx. The formula $x \mapsto \operatorname{ind}(u, x)$ determines a locally constant function on $\mathbb{R}^2 \smallsetminus u(S^1)$.

37.12x. Let $f : S^1 \to \mathbb{R}^2$ be a loop and let $x, y \in \mathbb{R}^2 \smallsetminus f(S^1)$. Prove that if $\operatorname{ind}(f, x) \neq \operatorname{ind}(f, y)$, then any path connecting x and y in $\mathbb{R}^2$ meets $f(S^1)$.

37.13x. Prove that if $u(S^1)$ is contained in a disk, while a point x is not, then $\operatorname{ind}(u, x) = 0$.

37.14x. Find the set of values of function $\operatorname{ind} : \mathbb{R}^2 \smallsetminus u(S^1) \to \mathbb{Z}$ for the following loops u:
a) $u(z) = z$; b) $u(z) = \bar{z}$; c) $u(z) = z^2$; d) $u(z) = z + z^{-1} + z^2 - z^{-2}$
(here $z \in S^1 \subset \mathbb{C}$).

37.15x. Choose several loops $u : S^1 \to \mathbb{R}^2$ such that $u(S^1)$ is a bouquet of two circles (a "lemniscate"). Find the winding number with respect to these loops for various points.

37.16x. Find a loop $f : S^1 \to \mathbb{R}^2$ such that there exist points $x, y \in \mathbb{R}^2 \smallsetminus f(S^1)$ with $\operatorname{ind}(f, x) = \operatorname{ind}(f, y)$, but belonging to different connected components of $\mathbb{R}^2 \smallsetminus f(S^1)$.

37.17x. Prove that any ray R radiating from x meets $f(S^1)$ at least at $|\operatorname{ind}(f, x)|$ points (i.e., the number of points in $f^{-1}(R)$ is at least $|\operatorname{ind}(f, x)|$).

37.Mx. *If* $u : S^1 \to \mathbb{R}^2$ *is a restriction of a continuous map* $F : D^2 \to \mathbb{R}^2$ *and* $\operatorname{ind}(u, x) \neq 0$*, then* $x \in F(D^2)$.

37.Nx. *If* u *and* v *are two circular loops in* $\mathbb{R}^2$ *with common base point (i. e.,* $u(1) = v(1)$*) and* uv *is their product, then* $\operatorname{ind}(uv, x) = \operatorname{ind}(u, x) + \operatorname{ind}(v, x)$ *for each* $x \in \mathbb{R}^2 \smallsetminus uv(S^1)$.

37.Ox. *Let u and v be circular loops in $\mathbb{R}^2$, and let $x \in \mathbb{R}^2 \smallsetminus (u(S^1) \cup v(S^1))$. If u and v are connected by a (free) homotopy u_t, $t \in I$ such that $x \in \mathbb{R}^2 \smallsetminus u_t(S^1)$ for each $t \in I$, then $\operatorname{ind}(u, x) = \operatorname{ind}(v, x)$.*

37.Px. *Let $u : S^1 \to \mathbb{C}$ be a circular loop, $a \in \mathbb{C}^2 \smallsetminus u(S^1)$. Then we have*

$$\operatorname{ind}(u, a) = \frac{1}{2\pi i} \int_{S^1} \frac{|u(z) - a|}{u(z) - a}\, dz.$$

37.Qx. *Let $p(z)$ be a polynomial with complex coefficients, let $R > 0$, and let $z_0 \in \mathbb{C}$. Consider the circular loop $u : S^1 \to \mathbb{C} : z \mapsto p(Rz)$. If $z_0 \in \mathbb{C} \smallsetminus u(S^1)$, then the polynomial $p(z) - z_0$ has (counting the multiplicities) precisely $\operatorname{ind}(u, z_0)$ roots in the open disk $B^2_R = \{z : |z| < R\}$.*

37.Rx. ***Riddle.*** By what can we replace the circular loop u, the domain B_R, and the polynomial $p(z)$ so that the assertion remains valid?

⌈37′5x⌋ Borsuk–Ulam Theorem

37.Sx One-Dimensional Borsuk–Ulam. For each continuous map $f : S^1 \to \mathbb{R}^1$, there exists $x \in S^1$ such that $f(x) = f(-x)$.

37.Tx Two-Dimensional Borsuk–Ulam. For each continuous map $f : S^2 \to \mathbb{R}^2$, there exists $x \in S^2$ such that $f(x) = f(-x)$.

37.Tx.1 Lemma. If there exists a continuous map $f : S^2 \to \mathbb{R}^2$ such that $f(x) \neq f(-x)$ for each $x \in S^2$, then there exists a continuous map $\varphi : \mathbb{R}P^2 \to \mathbb{R}P^1$ inducing a nonzero homomorphism $\pi_1(\mathbb{R}P^2) \to \pi_1(\mathbb{R}P^1)$.

37.18x. Prove that at each instant of time, there is a pair of antipodal points on the earth's surface where the pressures and also the temperatures are equal.

Theorems *37.Sx* and *37.Tx* are special cases of the following general theorem. We do not assume the reader is ready to prove Theorem *37.Ux* in the full generality, but is there another easy special case?

37.Ux Borsuk–Ulam Theorem. For each continuous map $f : S^n \to \mathbb{R}^n$, there exists $x \in S^n$ such that $f(x) = f(-x)$.

38. Retractions and Fixed Points

⌈38′1⌋ Retractions and Retracts

A continuous map of a topological space onto a subspace is a ***retraction*** if the restriction of the map to the subspace is the identity map. In other words, if X is a topological space and $A \subset X$, then $\rho : X \to A$ is a retraction if ρ is continuous and $\rho|_A = \mathrm{id}_A$.

38.A. Let ρ be a continuous map of a space X onto its subspace A. Then the following statements are equivalent:

(1) ρ is a retraction,

(2) $\rho(a) = a$ for any $a \in A$,

(3) $\rho \circ \mathrm{in} = \mathrm{id}_A$,

(4) $\rho : X \to A$ is an extension of the identity map $A \to A$.

A subspace A of a space X is a ***retract*** of X if there exists a retraction $X \to A$.

38.B. Any one-point subset is a retract.

A two-element set may be not a retract.

38.C. Any subset of $\mathbb{R}$ consisting of two points is not a retract of $\mathbb{R}$.

38.1. If A is a retract of X and B is a retract of A, then B is a retract of X.

38.2. If A is a retract of X and B is a retract of Y, then $A \times B$ is a retract of $X \times Y$.

38.3. A closed interval $[a, b]$ is a retract of $\mathbb{R}$.

38.4. An open interval (a, b) is not a retract of $\mathbb{R}$.

38.5. What topological properties of ambient space are inherited by a retract?

38.6. Prove that a retract of a Hausdorff space is closed.

38.7. Prove that the union of the Y axis and the set $\{(x, y) \in \mathbb{R}^2 \mid x > 0, y = \sin(1/x)\}$ is not a retract of $\mathbb{R}^2$ and, moreover, is not a retract of any of its neighborhoods.

38.D. S^0 is not a retract of D^1.

The role of the notion of retract is clarified by the following theorem.

38.E. *A subset A of a topological space X is a retract of X iff for each space Y each continuous map $A \to Y$ extends to a continuous map $X \to Y$.*

⌈38′2⌋ Fundamental Group and Retractions

38.F. *If $\rho : X \to A$ is a retraction, $i : A \to X$ is the inclusion, and $x_0 \in A$, then $\rho_* : \pi_1(X, x_0) \to \pi_1(A, x_0)$ is an epimorphism and $i_* : \pi_1(A, x_0) \to \pi_1(X, x_0)$ is a monomorphism.*

38.G. Riddle. Which of the two statements of Theorem *38.F* (about ρ_* or i_*) is easier to use for proving that a set $A \subset X$ is not a retract of X?

38.H Borsuk Theorem in Dimension 2. *S^1 is not a retract of D^2.*

38.8. Is the projective line a retract of the projective plane?

The following problem is more difficult than *38.H* in the sense that its solution is not a straightforward consequence of Theorem *38.F*, but rather demands to reexamine the arguments used in proof of *38.F*.

38.9. Prove that the boundary circle of the Möbius band is not a retract of the Möbius band.

38.10. Prove that the boundary circle of a handle is not a retract of the handle.

The Borsuk Theorem in its whole generality cannot be deduced like Theorem *38.H* from Theorem *38.F*. However, we can prove it by using a generalization of *38.F* to higher homotopy groups. Although we do not assume that you can successfully prove it now relying only on the tools provided above, we formulate it here.

38.I Borsuk Theorem. *The $(n-1)$-sphere S^{n-1} is not a retract of the n-disk D^n.*

At first glance this theorem seems to be useless. Why could it be interesting to know that a map with a very special property of being a retraction does not exist in this situation? However, in mathematics nonexistence theorems are often closely related to theorems that may seem to be more attractive. For instance, the Borsuk Theorem implies the Brouwer Theorem discussed below. But prior to this we must introduce an important notion related to the Brouwer Theorem.

⌈38′3⌋ Fixed-Point Property

Let $f : X \to X$ be a continuous map. A point $a \in X$ is a *fixed point* of f if $f(a) = a$. A space X has the *fixed-point property* if every continuous map $X \to X$ has a fixed point. The fixed point property implies solvability of a wide class of equations.

38.11. Prove that the fixed point property is a topological property.

38.12. A closed interval $[a, b]$ has the fixed point property.

38.13. Prove that if a topological space has the fixed point property, then so does each of its retracts.

38.14. Let X and Y be two topological spaces, $x_0 \in X$, and $y_0 \in Y$. Prove that X and Y have the fixed point property iff so does their bouquet $X \vee Y = X \sqcup Y/_{[x_0 \sim y_0]}$.

38.15. Prove that any finite tree has the fixed-point property. (We recall that a tree is a connected space obtained from a finite collection of closed intervals by somehow identifying their endpoints so that deleting an internal point from any of the segments makes the space disconnected, see 45′4x.) Is this statement true for infinite trees?

38.16. Prove that $\mathbb{R}^n$ with $n > 0$ does not have the fixed point property.

38.17. Prove that S^n does not have the fixed point property.

38.18. Prove that $\mathbb{R}P^n$ with odd n does not have the fixed point property.

38.19*. Prove that $\mathbb{C}P^n$ with odd n does not have the fixed point property.

Information. $\mathbb{R}P^n$ and $\mathbb{C}P^n$ with any even n have the fixed point property.

38.J Brouwer Theorem. *D^n has the fixed point property.*

38.J.1. Show that the Borsuk Theorem in dimension n (i.e., the statement that S^{n-1} is not a retract of D^n) implies the Brouwer Theorem in dimension n (i.e., the statement that any continuous map $D^n \to D^n$ has a fixed point).

38.K. Derive the Borsuk Theorem from the Brouwer Theorem.

The existence of fixed points can follow not only from topological arguments.

38.20. Prove that if $f : \mathbb{R}^n \to \mathbb{R}^n$ is a periodic affine transformation (i.e., $\underbrace{f \circ \dots \circ f}_{p \text{ times}} = \mathrm{id}_{\mathbb{R}^n}$ for a certain p), then f has a fixed point.

39. Homotopy Equivalences

⌈39′1⌋ Homotopy Equivalence as Map

Let X and Y be two topological spaces, and let $f : X \to Y$ and $g : Y \to X$ be continuous maps. Consider the compositions $f \circ g : Y \to Y$ and $g \circ f : X \to X$. They would be equal to the corresponding identity maps if f and g were mutually inverse homeomorphisms. If $f \circ g$ and $g \circ f$ are only homotopic to the identity maps, then f and g are *homotopy inverse* to each other. If a continuous map f possesses a homotopy inverse map, then f is a *homotopy invertible map* or a *homotopy equivalence*.

39.A. Prove the following properties of homotopy equivalences:

(1) any homeomorphism is a homotopy equivalence,
(2) a map homotopy inverse to a homotopy equivalence is a homotopy equivalence,
(3) the composition of two homotopy equivalences is a homotopy equivalence.

39.1. Find a homotopy equivalence that is not a homeomorphism.

⌈39′2⌋ Homotopy Equivalence as Relation

Two topological spaces X and Y are *homotopy equivalent* if there exists a homotopy equivalence $X \to Y$.

39.B. Homotopy equivalence of topological spaces is an equivalence relation.

The classes of homotopy equivalent spaces are *homotopy types*, and we say that homotopy equivalent spaces have the same homotopy type.

39.2. Prove that homotopy equivalent spaces have the same number of path-connected components.

39.3. Prove that homotopy equivalent spaces have the same number of connected components.

39.4. Find an infinite set of topological spaces that belong to the same homotopy type, but are pairwise non-homeomorphic.

⌈39′3⌋ Deformation Retraction

A retraction $\rho : X \to A$ is a *deformation retraction* if its composition $\text{in} \circ \rho$ with the inclusion $\text{in} : A \to X$ is homotopic to the identity id_X. If $\text{in} \circ \rho$ is A-homotopic to id_X, then ρ is a *strong deformation retraction*. If X admits a (strong) deformation retraction onto A, then A is a (*strong*) *deformation retract* of X.

39.C. Each deformation retraction is a homotopy equivalence.

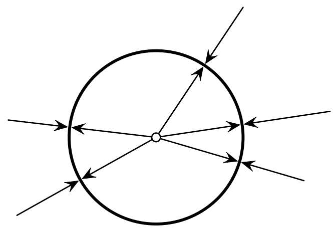
39.D. If A is a deformation retract of X, then A and X are homotopy equivalent.

39.E. Any two deformation retracts of one and the same space are homotopy equivalent.

39.F. If A is a deformation retract of X and B is a deformation retract of Y, then $A \times B$ is a deformation retract of $X \times Y$.

⌈39′4⌋ Examples

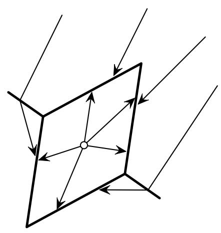
39.G. Circle S^1 is a deformation retract of $\mathbb{R}^2 \setminus 0$.



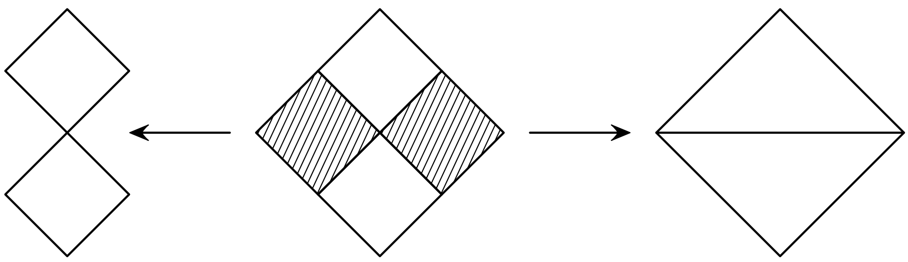
39.5. Prove that the Möbius strip is homotopy equivalent to a circle.

39.6. Classify letters of the Latin alphabet up to homotopy equivalence.

39.H. Prove that a plane with s punctures is homotopy equivalent to the union of s circles intersecting at a single point.



39.I. Prove that the union of a diagonal of a square and the contour of the same square is homotopy equivalent to the union of two circles intersecting at a single point.



39.7. Prove that a handle is homotopy equivalent to a bouquet of two circles. (E.g., construct a deformation retraction of the handle to the union of two circles intersecting at a single point.)

39.8. Prove that a handle is homotopy equivalent to the union of three arcs with common endpoints (i.e., the letter θ).

39.9. Prove that the space obtained from S^2 by identifying two (distinct) points is homotopy equivalent to the union of a two-sphere and a circle intersecting at a single point.

39.10. Prove that the space $\{(p,q) \in \mathbb{C} : z^2 + pz + q \text{ has two distinct roots}\}$ of quadratic complex polynomials with distinct roots is homotopy equivalent to the circle.

39.11. Prove that the space $GL(n,\mathbb{R})$ of invertible $n\times n$ real matrices is homotopy equivalent to the subspace $O(n)$ consisting of orthogonal matrices.

39.12. Riddle. Is there any relation between a solution of the preceding problem and the Gram–Schmidt orthogonalization? Can the Gram–Schmidt orthogonalization algorithm be regarded as a deformation retraction?

39.13. Construct the following deformation retractions: (a) $\mathbb{R}^3 \smallsetminus \mathbb{R}^1 \to S^1$; (b) $\mathbb{R}^n \smallsetminus \mathbb{R}^m \to S^{n-m-1}$; (c) $S^3 \smallsetminus S^1 \to S^1$; (d) $S^n \smallsetminus S^m \to S^{n-m-1}$ (e) $\mathbb{R}P^n \smallsetminus \mathbb{R}P^m \to \mathbb{R}P^{n-m-1}$.

⌈39′5⌋ Deformation Retraction versus Homotopy Equivalence

39.J. Spaces of Problem *39.I* cannot be embedded in one another. On the other hand, they can be embedded as deformation retracts in the plane with two punctures.

Deformation retractions constitute a very special class of homotopy equivalences. For example, they are often easier to visualize. However, as follows from *39.J*, it may happen that two spaces are homotopy equivalent, but none of them can be embedded in the other one, and so none of them is homeomorphic to a deformation retract of the other one. Therefore, deformation retractions seem to be insufficient for establishing homotopy equivalences.

However, this is not the case:

39.14*. Prove that any two homotopy equivalent spaces can be embedded as deformation retracts in the same topological space.

⌈39′6⌋ Contractible Spaces

A topological space X is *contractible* if the identity map $\mathrm{id} : X \to X$ is null-homotopic.

39.15. Show that $\mathbb{R}$ and I are contractible.

39.16. Prove that any contractible space is path-connected.

39.17. Prove that the following statements about a topological space X are equivalent:

(1) X is contractible,
(2) X is homotopy equivalent to a point,
(3) there exists a deformation retraction of X onto a point,
(4) each point a of X is a deformation retract of X,
(5) each continuous map of any topological space Y to X is null-homotopic,
(6) each continuous map of X to any topological space Y is null-homotopic.

39.18. Is it true that if X is a contractible space, then for any topological space Y

(1) any two continuous maps $X \to Y$ are homotopic?
(2) any two continuous maps $Y \to X$ are homotopic?

39.19. Find out if the spaces on the following list are contractible:

(1) $\mathbb{R}^n$,
(2) a convex subset of $\mathbb{R}^n$,
(3) a star-shaped subset of $\mathbb{R}^n$,
(4) $\{(x, y) \in \mathbb{R}^2 : x^2 - y^2 \le 1\}$,
(5) a finite tree (i.e., a connected space obtained from a finite collection of closed intervals by somehow identifying their endpoints so that deleting an internal point of each of the segments makes the space disconnected, see $45'4$x.)

39.20. Prove that $X \times Y$ is contractible iff both X and Y are contractible.

⌈39′7⌋ Fundamental Group and Homotopy Equivalences

39.K. *Let $f : X \to Y$ and $g : Y \to X$ be two homotopy inverse maps, and let $x_0 \in X$ and $y_0 \in Y$ be two points such that $f(x_0) = y_0$ and $g(y_0) = x_0$ and, moreover, the homotopies connecting $f \circ g$ with id_Y and $g \circ f$ with id_X are fixed at y_0 and x_0, respectively. Then f_* and g_* are mutually inverse isomorphisms between the groups $\pi_1(X, x_0)$ and $\pi_1(Y, y_0)$.*

39.L Corollary. *If $\rho : X \to A$ is a strong deformation retraction, $x_0 \in A$, then $\rho_* : \pi_1(X, x_0) \to \pi_1(A, x_0)$ and $\mathrm{in}_* : \pi_1(A, x_0) \to \pi_1(X, x_0)$ are mutually inverse isomorphisms.*

39.21. Calculate the fundamental group of the following spaces:
(a) $\mathbb{R}^3 \smallsetminus \mathbb{R}^1$,
(b) $\mathbb{R}^N \smallsetminus \mathbb{R}^n$,
(c) $\mathbb{R}^3 \smallsetminus S^1$,
(d) $\mathbb{R}^N \smallsetminus S^n$,
(e) $S^3 \smallsetminus S^1$,
(f) $S^N \smallsetminus S^k$,
(g) $\mathbb{R}P^3 \smallsetminus \mathbb{R}P^1$,
(h) handle,
(i) Möbius band,
(j) sphere with s holes,
(k) Klein bottle with a point removed,
(l) Möbius band with s holes.

39.22. Prove that the boundary circle of the Möbius band standardly embedded in $\mathbb{R}^3$ (see *22.18*) cannot be the boundary of a disk embedded in $\mathbb{R}^3$ in such a way that its interior does not meet the band.

39.23. 1) Calculate the fundamental group of the space Q of all complex polynomials $ax^2 + bx + c$ with distinct roots. 2) Calculate the fundamental group of the subspace Q_1 of Q consisting of polynomials with $a = 1$ (unitary polynomials).

39.24. ***Riddle.*** Can you solve *39.23* along the lines of deriving the customary formula for the roots of a quadratic trinomial?

39.M. Suppose that the assumptions of Theorem *39.K* are weakened as follows: $g(y_0) \neq x_0$ and/or the homotopies connecting $f \circ g$ with id_Y and $g \circ f$ with id_X are *not* fixed at y_0 and x_0, respectively. How would f_* and g_* be related? Would $\pi_1(X, x_0)$ and $\pi_1(Y, y_0)$ be isomorphic?

40. Covering Spaces via Fundamental Groups

⌈40′1⌋ Homomorphisms Induced by Covering Projections

40.A. Let $p : X \to B$ be a covering, $x_0 \in X$, and $b_0 = p(x_0)$. Then $p_* : \pi_1(X, x_0) \to \pi_1(B, b_0)$ is a monomorphism. Cf. *35.C*.

The image of the monomorphism $p_* : \pi_1(X, x_0) \to \pi_1(B, b_0)$ induced by the covering projection $p : X \to B$ is the *group of the covering* p *with base point* x_0.

40.B. Riddle. Is the group of the covering determined by the covering?

40.C Group of Covering versus Lifting of Loops. Let $p : X \to B$ be a covering. Describe the loops in B whose homotopy classes belong to the group of the covering in terms provided by Path Lifting Theorem *35.B*.

40.D. Let $p : X \to B$ be a covering, let $x_0, x_1 \in X$ belong to the same path-component of X, and let $b_0 = p(x_0) = p(x_1)$. Then $p_*(\pi_1(X, x_0))$ and $p_*(\pi_1(X, x_1))$ are conjugate subgroups of $\pi_1(B, b_0)$ (i.e., there is $\alpha \in \pi_1(B, b_0)$ such that $p_*(\pi_1(X, x_1)) = \alpha^{-1} p_*(\pi_1(X, x_0))\alpha$).

40.E. Let $p : X \to B$ be a covering, $x_0 \in X$, and $b_0 = p(x_0)$. For each $\alpha \in \pi_1(B, b_0)$, there exists an $x_1 \in p^{-1}(b_0)$ such that $p_*(\pi_1(X, x_1)) = \alpha^{-1} p_*(\pi_1(X, x_0))\alpha$.

40.F. Let $p : X \to B$ be a covering in the narrow sense, and let $G \subset \pi_1(B, b_0)$ be the group of this covering with a base point x_0. A subgroup $H \subset \pi_1(B, b_0)$ is a group of the same covering iff H is conjugate to G.

⌈40′2⌋ Number of Sheets

40.G Number of Sheets and Index of Subgroup. Let $p : X \to B$ be a finite-sheeted covering in the narrow sense. Then the number of sheets is equal to the index of the group of this covering.

40.H Sheets and Right Cosets. Let $p : X \to B$ be a covering in the narrow sense, $b_0 \in B$, and $x_0 \in p^{-1}(b_0)$. Construct a natural bijection between $p^{-1}(b_0)$ and the set $p_*(\pi_1(X, x_0))\backslash\pi_1(B, b_0)$ of right cosets of the group of the covering in the fundamental group of the base space.

40.1 Number of Sheets in Universal Covering. The number of sheets of a universal covering equals the order of the fundamental group of the base space.

40.2 Nontrivial Covering Means Nontrivial π_1. Any topological space that has a nontrivial path-connected covering space has a nontrivial fundamental group.

40.3. What numbers can appear as the number of sheets of a covering of the Möbius strip by the cylinder $S^1 \times I$?

40.4. What numbers can appear as the number of sheets of a covering of the Möbius strip by itself?

40.5. What numbers can appear as the number of sheets of a covering of the Klein bottle by a torus?

40.6. What numbers can appear as the number of sheets of a covering of the Klein bottle by itself?

40.7. What numbers can appear as the numbers of sheets for a covering of the Klein bottle by the plane $\mathbb{R}^2$?

40.8. What numbers can appear as the numbers of sheets for a covering of the Klein bottle by $S^1 \times \mathbb{R}$?

⌈40′3⌋ Hierarchy of Coverings

Let $p : X \to B$ and $q : Y \to B$ be two coverings, let $x_0 \in X$, $y_0 \in Y$, and $p(x_0) = q(y_0) = b_0$. The covering q with base point y_0 is ***subordinate*** to p with base point x_0 if there exists a map $\varphi : X \to Y$ such that $q \circ \varphi = p$ and $\varphi(x_0) = y_0$. In this case, the map φ is a ***subordination***.

40.I. A subordination is a covering map.

40.J. If a subordination exists, then it is unique. Cf. *35.B*.

Two coverings $p : X \to B$ and $q : Y \to B$ are ***equivalent*** if there exists a homeomorphism $h : X \to Y$ such that $p = q \circ h$. In this case, h and h^{-1} are ***equivalences***.

40.K. If two coverings are subordinated to each other, then the corresponding subordinations are equivalences.

40.L. The equivalence of coverings is, indeed, an equivalence relation on the set of coverings with a given base space.

40.M. Subordination determines a nonstrict partial order on the set of equivalence classes of coverings with a given base.

40.9. What equivalence class of coverings is minimal (i.e., subordinated to all other classes)?

40.N. Let $p : X \to B$ and $q : Y \to B$ be two coverings, and let $x_0 \in X$, $y_0 \in Y$, and $p(x_0) = q(y_0) = b_0$. If q with base point y_0 is subordinated to p with base point x_0, then the group of the covering p is contained in the group of the covering q, i.e., $p_*(\pi_1(X, x_0)) \subset q_*(\pi_1(Y, y_0))$.

41x. Classification of Covering Spaces

⌈41′1x⌋ Existence of Subordinations

A topological space X is *locally path-connected* if for each point $a \in X$ and each neighborhood U of a the point a has a path-connected neighborhood $V \subset U$.

41.1x. Find a topological space which is path-connected, but not locally path-connected.

41.Ax. Let B be a locally path-connected space, let $p : X \to B$ and $q : Y \to B$ be two coverings in the narrow sense, and let $x_0 \in X$, $y_0 \in Y$, and $p(x_0) = q(y_0) = b_0$. If $p_*(\pi_1(X, x_0)) \subset q_*(\pi_1(Y, y_0))$, then q is subordinated to p.

41.Ax.1. Under the conditions of *41.Ax*, if two paths $u, v : I \to X$ have the same initial point x_0 and a common final point, then the paths that cover $p \circ u$ and $p \circ v$ and have the same initial point y_0 also have the same final point.

41.Ax.2. Under the conditions of *41.Ax*, the map $X \to Y$ defined by *41.Ax.1* (guess what this map is!) is continuous.

41.2x. Construct an example proving that the hypothesis of local path connectedness in *41.Ax.2* and *41.Ax* is necessary.

41.Bx. Two coverings $p : X \to B$ and $q : Y \to B$ with a common locally path-connected base are equivalent iff for some $x_0 \in X$ and $y_0 \in Y$ with $p(x_0) = q(y_0) = b_0$ the groups $p_*(\pi_1(X, x_0))$ and $q_*(\pi_1(Y, y_0))$ are conjugate in $\pi_1(B, b_0)$.

41.3x. Construct an example proving that the assumption of local path connectedness of the base in *41.Bx* is necessary.

⌈41′2x⌋ Micro Simply Connected Spaces

A topological space X is *micro simply connected* if each point $a \in X$ has a neighborhood U such that the inclusion homomorphism $\pi_1(U, a) \to \pi_1(X, a)$ is trivial.

41.4x. Any simply connected space is micro simply connected.

41.5x. Find a micro simply connected, but not simply connected space.

A topological space is *locally contractible at point* a if each neighborhood U of a contains a neighborhood V of a such that the inclusion $V \to U$ is null-homotopic. A topological space is *locally contractible* if it is locally contractible at each of its points.

41.6x. Any finite topological space is locally contractible.

41.7x. Any locally contractible space is micro simply connected.

41.8x. Find a space which is not micro simply connected.

In the literature, the micro simply connectedness is also called *weak local simply connectedness*, while a strong local simply connectedness is the following property: any neighborhood U of any point x contains a neighborhood V such that any loop at x in V is null-homotopic in U.

41.9x. Find a micro simply connected space which is not strong locally simply connected.

⌈41′3x⌋ Existence of Coverings

41.Cx. *A space having a universal covering space is micro simply connected.*

41.Dx Existence of a Covering with a Given Group. *If a topological space B is path-connected, locally path-connected, and micro simply connected, then for any $b_0 \in B$ and any subgroup π of $\pi_1(B, b_0)$ there exists a covering $p : X \to B$ and a point $x_0 \in X$ such that $p(x_0) = b_0$ and $p_*(\pi_1(X, x_0)) = \pi$.*

41.Dx.1. Suppose that in the assumptions of Theorem *41.Dx* there exists a covering $p : X \to B$ satisfying all requirements of this theorem. For each $x \in X$, describe all paths in B that are p-images of paths connecting x_0 to x in X.

41.Dx.2. Does the solution of Problem *41.Dx.1* determine an equivalence relation on the set of all paths in B starting at b_0, so that we obtain a one-to-one correspondence between the set X and the set of equivalence classes?

41.Dx.3. Describe a topology on the set of equivalence classes from *41.Dx.2* such that the natural bijection between X and this set is a homeomorphism.

41.Dx.4. Prove that the reconstruction of X and $p : X \to B$ provided by Problems *41.Dx.1*–*41.Dx.4* under the assumptions of Theorem *41.Dx* determine a covering whose existence is claimed by Theorem *41.Dx*.

Essentially, assertions *41.Dx.1* - *41.Dx.3* imply the uniqueness of the covering with a given group. More precisely, the following assertion holds true.

41.Ex Uniqueness of the Covering with a Given Group. *Assume that B is path-connected, locally path-connected, and micro simply connected. Let $p : X \to B$ and $q : Y \to B$ be two coverings, and let $p_*(\pi_1(X, x_0)) = q_*(\pi_1(Y, y_0))$. Then the coverings p and q are equivalent, i.e., there exists a homeomorphism $f : X \to Y$ such that $f(x_0) = y_0$ and $p \circ f = q$.*

41.Fx Classification of Coverings over a Good Space. *Let B be a path-connected, locally path-connected, and micro simply connected space with base point b_0. Then there is a one-to-one correspondence between classes of equivalent coverings (in the narrow sense) over B and conjugacy classes of subgroups of $\pi_1(B, b_0)$. This correspondence identifies the hierarchy of coverings (ordered by subordination) with the hierarchy of subgroups (ordered by inclusion).*

Under the correspondence of Theorem *41.Fx*, the trivial subgroup corresponds to a covering with simply connected covering space. Since this covering subordinates any other covering with the same base space, it is said to be *universal*.

41.10x. Describe all coverings of the following spaces up to equivalence and subordination:

(1) circle S^1;
(2) punctured plane $\mathbb{R}^2 \smallsetminus 0$;
(3) Möbius strip;
(4) four-point digital circle (the space formed by 4 points, a, b, c, d; with the base of open sets formed by $\{a\}$, $\{c\}$, $\{a, b, c\}$, and $\{c, d, a\}$)
(5) torus $S^1 \times S^1$;

⌈41′4x⌋ Action of Fundamental Group on Fiber

41.Gx Action of π_1 on Fiber. Let $p : X \to B$ be a covering, $b_0 \in B$. Construct a natural right action of $\pi_1(B, b_0)$ on $p^{-1}(b_0)$.

41.Hx. When the action in *41.Gx* is transitive?

⌈41′5x⌋ Automorphisms of Covering

A homeomorphism $\varphi : X \to X$ is an *automorphism* of a covering $p : X \to B$ if $p \circ \varphi = p$.

41.Ix. Automorphisms of a covering form a group.

We denote the group of automorphisms of a covering $p : X \to B$ by $\mathrm{Aut}(p)$.

41.Jx. *An automorphism $\varphi : X \to X$ of the covering $p : X \to B$ is determined by the image $\varphi(x_0)$ of any $x_0 \in X$. Cf. 40.J.*

41.Kx. *Any two-fold covering has a nontrivial automorphism.*

41.11x. Find a three-fold covering without nontrivial automorphisms.

Let G be a group and H its subgroup. Recall that the *normalizer* $N(H)$ of H is the subset of G consisting of $g \in G$ such that $g^{-1}Hg = H$. This is a subgroup of G, which contains H as a normal subgroup. So, $N(H)/_H$ is a group.

41.Lx. Let $p : X \to B$ be a covering, $x_0 \in X$ and $b_0 = p(x_0)$. Construct a map $\pi_1(B, b_0) \to p^{-1}(b_0)$ which induces a bijection of the set $p_*(\pi_1(X, x_0))\backslash\pi_1(B, b_0)$ of right cosets onto $p^{-1}(b_0)$.

41.Mx. Show that the bijection $p_*(\pi_1(X, x_0))\backslash\pi_1(B, b_0) \to p^{-1}(b_0)$ constructed in *41.Lx* maps the set of images of x_0 under all automorphisms of a covering $p : X \to B$ to the group $N(p_*(\pi_1(X, x_0)))/_{p_*(\pi_1(X,x_0))}$.

41.Nx. *For any covering $p : X \to B$ in the narrow sense, there is a natural injective map $Aut(p)$ to the group $N(p_*(\pi_1(X, x_0)))/_{p_*(\pi_1(X,x_0))}$. This map is an antihomomorphism.*[1]

41.Ox. *Under assumptions of Theorem 41.Nx, if B is locally path-connected, then the antihomomorphism $Aut(p) \to N(p_*(\pi_1(X, x_0)))/p_*(\pi_1(X, x_0))$ is bijective.*

⌈41′6x⌋ Regular Coverings

41.Px Regularity of Covering. *Let $p : X \to B$ be a covering in the narrow sense, $b_0 \in B$, and $x_0 \in p^{-1}(b_0)$. The following conditions are equivalent:*

(1) *$p_*\big(\pi_1(X, x_0)\big)$ is a normal subgroup of $\pi_1(B, b_0)$;*
(2) *$p_*\big(\pi_1(X, x)\big)$ is a normal subgroup of $\pi_1(B, p(x))$ for each $x \in X$;*
(3) *all groups $p_*\pi_1(X, x)$ for $x \in p^{-1}(b)$ are the same;*
(4) *for each loop $s : I \to B$, either every path in X covering s is a loop (independently of the initial point), or none of them is a loop;*
(5) *the automorphism group acts transitively on $p^{-1}(b_0)$.*

A covering satisfying to (any of) the equivalent conditions of Theorem *41.Px* is said to be *regular*. Otherwise, the covering is *irregular*.

41.12x. The coverings $\mathbb{R} \to S^1 : x \mapsto e^{2\pi i x}$ and $S^1 \to S^1 : z \mapsto z^n$ for integer $n > 0$ are regular.

41.Qx. *The automorphism group of a regular covering $p : X \to B$ is naturally anti-isomorphic to the quotient group $\pi_1(B, b_0)/_{p_*\pi_1(X,x_0)}$ of the group $\pi_1(B, b_0)$ by the group of the covering for any $x_0 \in p^{-1}(b_0)$.*

41.Rx Classification of Regular Coverings over a Good Base. *There is a one-to-one correspondence between classes of equivalent coverings (in the narrow sense) over a path-connected, locally path-connected, and micro simply connected space B with a base point b_0, on one hand, and anti-epimorphisms $\pi_1(B, b_0) \to G$, on the other hand.*

[1] Recall that a map $\varphi : G \to H$ from a group G to a group H is an *antihomomorphism* if $\varphi(ab) = \varphi(b)\varphi(a)$ for any $a, b \in G$.

Algebraic properties of the automorphism group of a regular covering are often referred to as if they were properties of the covering itself. For instance, a *cyclic covering* is a regular covering with cyclic automorphism group, an *Abelian covering* is a regular covering with Abelian automorphism group, etc.

41.13x. Any two-fold covering is regular.

41.14x. Which coverings considered in the problems of Section 34 are regular? Are there any irregular coverings?

41.15x. Find a three-fold irregular covering of a bouquet of two circles.

41.16x. Let $p : X \to B$ be a regular covering, $Y \subset X$, and $C \subset B$, and let $q : Y \to C$ be a submap of p. Prove that if q is a covering, then this covering is regular.

⌈41′7x⌋ Lifting and Covering Maps

41.Sx. Riddle. Let $p : X \to B$ and $f : Y \to B$ be continuous maps. Let $x_0 \in X$ and $y_0 \in Y$ be points such that $p(x_0) = f(y_0)$. In terms of the homomorphisms $p_* : \pi_1(X, x_0) \to \pi_1(B, p(x_0))$ and $f_* : \pi_1(Y, y_0) \to \pi_1(B, f(y_0))$, formulate a necessary condition for f to have a lift $\widetilde{f} : Y \to X$ such that $\widetilde{f}(y_0) = x_0$. Find an example in which this condition is not sufficient. What additional assumptions can make it sufficient?

41.Tx Theorem on Lifting a Map. *Let $p : X \to B$ be a covering in the narrow sense and $f : Y \to B$ be a continuous map. Let $x_0 \in X$ and $y_0 \in Y$ be points such that $p(x_0) = f(y_0)$. If Y is a locally path-connected space and $f_*\pi(Y, y_0) \subset p_*\pi(X, x_0)$, then there exists a unique continuous map $\widetilde{f} : Y \to X$ such that $p \circ \widetilde{f} = f$ and $\widetilde{f}(y_0) = x_0$.*

41.Ux. *Let $p : X \to B$ and $q : Y \to C$ be two coverings in the narrow sense, and let $f : B \to C$ be a continuous map. Let $x_0 \in X$ and $y_0 \in Y$ be points such that $fp(x_0) = q(y_0)$. If there exists a continuous map $F : X \to Y$ such that $fp = qF$ and $F(x_0) = y_0$, then we have $f_*p_*\pi_1(X, x_0) \subset q_*\pi_1(Y, y_0)$.*

41.Vx Theorem on Covering of a Map. *Let $p : X \to B$ and $q : Y \to C$ be two coverings in the narrow sense, $f : B \to C$ a continuous map. Let $x_0 \in X$ and $y_0 \in Y$ be points such that $fp(x_0) = q(y_0)$. If Y is locally path-connected and $f_*p_*\pi_1(X, x_0) \subset q_*\pi_1(Y, y_0)$, then there exists a unique continuous map $F : X \to Y$ such that $fp = qF$ and $F(x_0) = y_0$.*

⌈41′8x⌋ Induced Coverings

41.Wx. *Let $p : X \to B$ be a covering, $f : A \to B$ a continuous map. Denote by W the subspace of $A \times X$ consisting of points (a, x) such that $f(a) = p(x)$. Let $q : W \to A$ be the restriction of the projection $A \times X \to A$. Then $q : W \to A$ is a covering with the same number of sheets as p.*

A covering $q : W \to A$ obtained as in Theorem *41.Wx* is said to be *induced* from $p : X \to B$ by $f : A \to B$.

41.17x. Represent coverings from Problems *34.D* and *34.F* as coverings induced from $\mathbb{R} \to S^1 : x \mapsto e^{2\pi i x}$.

41.18x. Which of the coverings considered above is induced from the covering of Problem *36.7*?

⌈41′9x⌋ High-Dimensional Homotopy Groups of Covering Space

41.Xx. *Let $p : X \to B$ be a covering. Then for any continuous map $s : I^n \to B$ and any lift $u : I^{n-1} \to X$ of the restriction $s|_{I^{n-1}}$ the map s has a unique lift extending u.*

41.Yx. *For any covering $p : X \to B$ and points $x_0 \in X$ and $b_0 \in B$ such that $p(x_0) = b_0$, the homotopy groups $\pi_r(X, x_0)$ and $\pi_r(B, b_0)$ with $r > 1$ are canonically isomorphic.*

41.Zx. Prove that homotopy groups of dimensions greater than 1 of circle, torus, Klein bottle and Möbius strip are trivial.

Chapter IX

Cellular Techniques

42. Cellular Spaces

⌈42′1⌋ Definition of Cellular Spaces

In this section, we study a class of topological spaces that play a very important role in algebraic topology. Their role in the context of this book is more restricted: this is the class of spaces for which we learn how to calculate the fundamental group.[1]

A *zero-dimensional cellular space* is just a discrete space. Points of a 0-dimensional cellular space are also called (*zero-dimensional*) *cells*, or 0-*cells*.

A *one-dimensional cellular space* is a space that can be obtained as follows. Take any 0-dimensional cellular space X_0. Take a family of maps $\varphi_\alpha : S^0 \to X_0$. Attach the sum of a family of copies of D^1 to X_0 via φ_α (the copies are indexed by the same indices α as the maps φ_α):

$$X_0 \cup_{\sqcup\varphi_\alpha} \Big(\bigsqcup_\alpha D^1\Big).$$

The images of copies of the interior parts $\operatorname{Int} D^1$ of D^1 are called (*open*) 1-*dimensional cells*, 1-*cells*, *one-cells*, or *edges*. The subsets obtained from D^1 are *closed* 1-*cells*. The cells of X_0 (i.e., points of X_0) are also called *vertices*.

[1] This class of spaces was introduced by J. H. C. Whitehead. He called these spaces *CW-complexes*, and they are known under this name. However, it is not a good name for plenty of reasons. With very rare exceptions (one of which is *CW*-complex, the other is simplicial complex), the word *complex* is used nowadays for various algebraic notions, but not for spaces. We have decided to use the term *cellular space* instead of *CW-complex* following D. B. Fuchs and V. A. Rokhlin [**2**].

Open 1-cells and 0-cells constitute a partition of a one-dimensional cellular space. This partition is included in the notion of cellular space. In other words, a one-dimensional cellular space is a topological space equipped with a partition that can be obtained in this way.[2]

A ***two-dimensional cellular space*** is a space that can be obtained as follows. Take any cellular space X_1 of dimension 0 or 1. Take a family of continuous[3] maps $\varphi_\alpha : S^1 \to X_1$. Attach the sum of a family of copies of D^2 to X_1 via φ_α:

$$X_1 \cup_{\sqcup\varphi_\alpha} \Big(\bigsqcup_\alpha D^2\Big).$$

The images of the interior parts of copies of D^2 are (***open***) ***2-dimensional cells***, ***2-cells***, ***two-cells***, or ***faces***. The cells of X_1 are also regarded as cells of the 2-dimensional cellular space. Open cells of both kinds constitute a partition of a 2-dimensional cellular space. This partition is included in the notion of cellular space, i.e., a two-dimensional cellular space is a topological space equipped with a partition that can be obtained in the way described above. The set obtained out of a copy of the whole D^2 is a ***closed 2-cell***.

A ***cellular space of dimension*** n is defined in a similar way: This is a space equipped with a partition. It is obtained from a cellular space X_{n-1} of dimension less than n by attaching a family of copies of the n-disk D^n via a family of continuous maps of their boundary spheres:

$$X_{n-1} \cup_{\sqcup\varphi_\alpha} \Big(\bigsqcup_\alpha D^n\Big).$$

The images of the interiors of the attached n-disks are (***open***) ***n-dimensional cells*** or simply ***n-cells***. The images of the entire n-disks are ***closed n-cells***. Cells of X_{n-1} are also regarded as cells of the n-dimensional cellular space.

[2] One-dimensional cellular spaces are also associated with the word *graph*. However, rather often, this word is used for objects of other classes. For example, one can call in this way one-dimensional cellular spaces in which attaching maps of different one-cells cannot coincide, or the boundaries of one-cells cannot consist of a single vertex. When one-dimensional cellular spaces are to be considered anyway, inspite of this terminological disregard, they are called *multigraphs* or *pseudographs*. Furthermore, sometimes one includes an additional structure into the notion of graph—say, a choice of orientation on each edge. Certainly, all of these variations contradict a general tendency in mathematical terminology to give simple names to decent objects of a more general nature, passing to more complicated terms while adding structures and imposing restrictions. However, in this specific situation there is no hope to implement that tendency. Any attempt to fix a meaning for the word *graph* apparently only contributes to this chaos, and we just keep this word away from important formulations, using it as a short informal synonym for the more formal term of one-dimensional cellular space. (Other overused common words, like *curve* and *surface*, also deserve this sort of caution.)

[3] In the above definition of a 1-dimensional cellular space, the attaching maps φ_α were also continuous, although their continuity was not required since any map of S^0 to any space is continuous.

Each of the mappings φ_α is an ***attaching map***, and the restriction of the corresponding factorization map to the n-disk D^n is the ***characteristic map***.

A ***cellular space*** is obtained as the union of an increasing sequence of cellular spaces $X_0 \subset X_1 \subset \dots \subset X_n \subset \dots$ obtained in this way from each other. The sequence may be finite or infinite. In the latter case, the topological structure is introduced by saying that the cover of the union by the sets X_n is fundamental, i.e., a set $U \subset \bigcup_{n=0}^{\infty} X_n$ is open iff its intersection $U \cap X_n$ with each X_n is open in X_n.

The partition of a cellular space into its open cells is a ***cellular decomposition***. The union of all cells of dimension less than or equal to n of a cellular space X is the ***n-dimensional skeleton*** of X. This term may be misleading since the n-dimensional skeleton may contain no n-cells, and so it may coincide with the $(n-1)$-dimensional skeleton. Thus, the n-dimensional skeleton may have dimension less than n. For this reason, it is better to speak about the ***nth skeleton*** or ***n-skeleton***.

42.1. In a cellular space, skeletons are closed.

A cellular space is ***finite*** if it contains a finite number of cells. A cellular space is ***countable*** if it contains a countable number of cells. A cellular space is ***locally finite*** if each of its points has a neighborhood that meets finitely many cells.

Let X be a cellular space. A subspace $A \subset X$ is a ***cellular subspace*** of X if A is a union of open cells and together with each cell e contains the closed cell $\bar{e}$. This definition admits various equivalent reformulations. For instance, $A \subset X$ is a ***cellular subspace*** of X iff A is both a union of closed cells and a union of open cells. Another option: together with each point $x \in A$ the subspace A contains the closed cell $\overline{e} \in x$. Certainly, A is equipped with a partition into the open cells of X contained in A. Obviously, the k-skeleton of a cellular space X is a cellular subspace of X.

42.2. Prove that the union and intersection of any collection of cellular subspaces are cellular subspaces.

42.A. Prove that a cellular subspace of a cellular space is a cellular space. (Probably, your proof will involve assertion *43.Fx*.)

42.A.1. Let X be a topological space, and let $X_1 \subset X_2 \subset \dots$ be an increasing sequence of subsets constituting a fundamental cover of X. Let $A \subset X$ be a subspace; denote $A \cap X_i$ by A_i. Let one of the following conditions be fulfilled:
1) X_i is open in X for each i;
2) A_i is open in X for each i;
3) A_i is closed in X for each i.
Then $\{A_i\}$ is a fundamental cover of A.

⌈42′2⌋ First Examples

42.B. A cellular space consisting of two cells, where one is a 0-cell and the other one is an n-cell, is homeomorphic to S^n.

42.C. Represent D^n with $n > 0$ as a cellular space made of three cells.

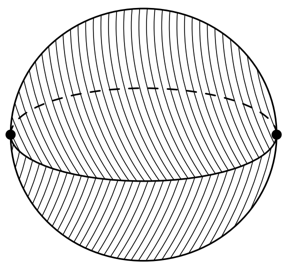
42.D. A cellular space consisting of a single 0-cell and q one-cells is a bouquet of q circles.

42.E. Represent torus $S^1 \times S^1$ as a cellular space with one 0-cell, two 1-cells, and one 2-cell.

42.F. How would you obtain a presentation of torus $S^1 \times S^1$ as a cellular space with 4 cells from a presentation of S^1 as a cellular space with 2 cells?

> ***42.3.*** Prove that if X and Y are finite cellular spaces, then $X \times Y$ has a natural structure of a finite cellular space.
>
> ***42.4*.*** Does the statement of Problem *42.3* remain true if we skip the finiteness condition in it? If yes, prove this; if no, find an example in which the product is not a cellular space.

42.G. Represent the sphere S^n as a cellular space such that the spheres $S^0 \subset S^1 \subset S^2 \subset \dots \subset S^{n-1}$ are its skeletons.



42.H. Represent $\mathbb{R}P^n$ as a cellular space with $n + 1$ cells. Describe the attaching maps of the cells.

> ***42.5.*** Represent $\mathbb{C}P^n$ as a cellular space with $n+1$ cells. Describe the attaching maps of its cells.
>
> ***42.6.*** Represent the following topological spaces as cellular ones
> (a) handle; (b) Möbius strip; (c) $S^1 \times I$,
> (d) sphere with p handles; (e) sphere with p cross-caps.
>
> ***42.7.*** What is the minimal number of cells in a cellular space homeomorphic to
> (a) Möbius strip; (b) sphere with p handles; (c) sphere with p cross-caps?
>
> ***42.8.*** Find a cellular space where the closure of a cell is not equal to a union of other cells. What is the minimal number of cells in a space containing a cell of this sort?

42.9. Consider the disjoint sum of countably many copies of the closed interval I and identify the copies of 0 in all of them. Represent the result (which is the bouquet of the countable family of intervals) as a countable cellular space. Prove that this space is not first countable.

42.I. Represent $\mathbb{R}^1$ as a cellular space.

42.10. Prove that for any two cellular spaces homeomorphic to $\mathbb{R}^1$ there exists a homeomorphism between them which homeomorphically maps each cell of one of them onto a cell of the other one.

42.J. Represent $\mathbb{R}^n$ as a cellular space.

Denote by $\mathbb{R}^\infty$ the union of the sequence of Euclidean spaces $\mathbb{R}^0 \subset \mathbb{R}^1 \subset \dots \subset \mathbb{R}^n \subset$ canonically included to each other: $\mathbb{R}^n = \{x \in \mathbb{R}^{n+1} : x_{n+1} = 0\}$. Equip $\mathbb{R}^\infty$ with the topological structure for which the spaces $\mathbb{R}^n$ constitute a fundamental cover.

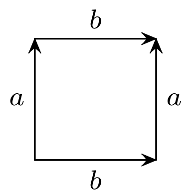
42.K. Represent $\mathbb{R}^\infty$ as a cellular space.

42.11. Show that $\mathbb{R}^\infty$ is not metrizable.

⌈42′3⌋ Further Two-Dimensional Examples

We consider a class of 2-dimensional cellular spaces that admit a simple combinatorial description. Each space in this class is a quotient space of a finite family of convex polygons by identification of sides via affine homeomorphisms. The identification of vertices is determined by the identification of the sides. The quotient space has a natural decomposition into 0-cells, which are the images of vertices; 1-cells, which are the images of sides; and faces, which are the images of the interior parts of the polygons.

To describe such a space, we first need to show what sides are identified. Usually this is indicated by writing the same letters at the sides to be identified. There are only two affine homeomorphisms between two closed intervals. To specify one of them, it suffices to show the orientations of the intervals that are identified by the homeomorphism. Usually this is done by drawing arrows on the sides. Here is a description of this sort for the standard presentation of torus $S^1 \times S^1$ as the quotient space of square:



We can replace a picture by a combinatorial description. To do this, put letters on *all* sides of the polygon, go around the polygons counterclockwise and write down the letters that stay at the sides of polygon along the contour. The letters corresponding to the sides whose orientation is opposite to the counterclockwise direction are put with exponent -1. This yields a collection of words, which contains sufficient information about the family of polygons and the partition. For instance, the presentation of the torus shown above is encoded by the word $ab^{-1}a^{-1}b$.

42.12. Prove that:

(1) the word $a^{-1}a$ describes a cellular space homeomorphic to S^2,
(2) the word aa describes a cellular space homeomorphic to $\mathbb{R}P^2$,
(3) the word $aba^{-1}b^{-1}c$ describes a handle,
(4) the word $abcb^{-1}$ describes cylinder $S^1 \times I$,
(5) each of the words aab and $abac$ describe Möbius strip,
(6) the word $abab$ describes a cellular space homeomorphic to $\mathbb{R}P^2$,
(7) each of the words $aabb$ and $ab^{-1}ab$ describe Klein bottle,
(8) the word

$$a_1b_1a_1^{-1}b_1^{-1}a_2b_2a_2^{-1}b_2^{-1}\dots a_gb_ga_g^{-1}b_g^{-1}$$

describes sphere with g handles,
(9) the word $a_1a_1a_2a_2\dots a_ga_g$ describes sphere with g cross-caps.

⌈42′4⌋ Embedding in Euclidean Space

42.L. Any countable 0-dimensional cellular space can be embedded in $\mathbb{R}$.

42.M. Any countable locally finite 1-dimensional cellular space can be embedded in $\mathbb{R}^3$.

42.13. Find a 1-dimensional cellular space which you cannot embed in $\mathbb{R}^2$. (We do not ask you to prove rigorously that no embedding is possible.)

42.N. *Any finite dimensional countable locally finite cellular space can be embedded in a Euclidean space of sufficiently high dimension.*

42.N.1. Let X and Y be topological spaces such that X can be embedded in $\mathbb{R}^p$, Y can be embedded in $\mathbb{R}^q$, and both embeddings are proper maps. (See 19′3x; in particular, their images are closed in $\mathbb{R}^p$ and $\mathbb{R}^q$, respectively.) Let A be a closed subset of Y. Assume that A has a neighborhood U in Y such that there exists a homeomorphism $h : \operatorname{Cl} U \to A \times I$ mapping A to $A \times 0$. Let $\varphi : A \to X$ be a proper continuous map. Then the initial embedding $X \to \mathbb{R}^p$ extends to an embedding $X \cup_\varphi Y \to \mathbb{R}^{p+q+1}$.

42.N.2. Let X be a locally finite countable k-dimensional cellular space, A the $(k-1)$-skeleton of X. Prove that if A can be embedded in $\mathbb{R}^p$, then X can be embedded in $\mathbb{R}^{p+k+1}$.

42.O. Any countable locally finite cellular space can be embedded in $\mathbb{R}^\infty$.

42.P. Any finite cellular space is metrizable.

42.Q. Any finite cellular space is normal.

42.R. Any countable cellular space can be embedded in $\mathbb{R}^\infty$.

42.S. Any cellular space is normal.

42.T. Any locally finite cellular space is metrizable.

⌈42′5x⌋ Simplicial Spaces

Recall that in 24′3x we introduced a class of topological spaces: simplicial spaces. Each simplicial space is equipped with a partition into subsets, called open simplices, which are indeed homeomorphic to open simplices of Euclidean space.

42.Ux. Any simplicial space is cellular, and its partition into open simplices is the corresponding partition into open cells.

43x. Topological Properties of Cellular Spaces

The present section contains assertions of mixed character. For example, we study conditions ensuring that a cellular space is compact (*43.J*x) or separable (*43.N*x). We also prove that a cellular space X is connected, iff X is path-connected (*43.R*x), iff the 1-skeleton of X is path-connected (*43.U*x). On the other hand, we study the cellular topological structure as such. For example, any cellular space is Hausdorff (*43.A*x). Further, it is not clear at all from the definition of a cellular space that a closed cell is the closure of the corresponding open cell (or that closed cells are closed sets). In this connection, the present section includes assertions of technical character. (We do not formulate them as lemmas to individual theorems because often they are lemmas for several assertions.) For example: closed cells constitute a fundamental cover of a cellular space (*43.C*x).

We notice that in textbooks (say, in the textbook [**2**] by Fuchs and Rokhlin) a cellular space is defined as a Hausdorff topological space equipped by a cellular partition with two properties:
(C) each closed cell meets only a finite number of (open) cells;
(W) closed cells constitute a fundamental cover of the space.

The results of assertions *43.A*x, *43.B*x, and *43.E*x imply that cellular spaces in the sense of the above definition are cellular spaces in the sense of Fuchs - Rokhlin' textbook (i.e., in the standard sense), the possibility of inductive construction for which is proved in [**2**]. Thus, both definitions of a cellular space are equivalent.

An advice to the reader: first try to prove the above assertions for finite cellular spaces.

***43.A*x.** Each cellular space is a Hausdorff topological space.

***43.B*x.** In a cellular space, the closure of any cell e is the closed cell $\overline{e}$.

***43.C*x.** Closed cells constitute a fundamental cover of a cellular space.

***43.D*x.** *Each cover of a cellular space by cellular subspaces is fundamental.*

***43.E*x.** In a cellular space, any closed cell meets only a finite number of open cells.

***43.F*x.** If A is cellular subspace of a cellular space X, then A is closed in X.

***43.G*x.** The space obtained as a result of pasting two cellular subspaces together along their common subspace, is cellular.

43.Hx. If a subset A of a cellular space X intersects each open cell along a finite set, then A is closed. Furthermore, the induced topology on A is discrete.

43.Ix. Prove that each compact subset of a cellular space meets a finite number of cells.

43.Jx Corollary. *A cellular space is compact iff it is finite.*

43.Kx. Any cell of a cellular space is contained in a finite cellular subspace of this space.

43.Lx. Any compact subset of a cellular space is contained in a finite cellular subspace.

43.Mx. *A subset of a cellular space is compact iff it is closed and meets only a finite number of open cells.*

43.Nx. A cellular space is separable iff it is countable.

43.Ox. Any path-connected component of a cellular space is a cellular subspace.

43.Px. A cellular space is locally path-connected.

43.Qx. Any path-connected component of a cellular space is both open and closed. It is a connected component.

43.Rx. A cellular space is connected iff it is path-connected.

43.Sx. A locally finite cellular space is countable iff it has countable 0-skeleton.

43.Tx. Any connected locally finite cellular space is countable.

43.Ux. *A cellular space is connected iff its* 1*-skeleton is connected.*

44. Cellular Constructions

⌈44′1⌋ Euler Characteristic

Let X be a finite cellular space. Let $c_i(X)$ denote the number of its cells of dimension i. The ***Euler characteristic*** of X is the alternating sum of $c_i(X)$:

$$\chi(X) = c_0(X) - c_1(X) + c_2(X) - \dots + (-1)^i c_i(X) + \dots .$$

44.A. Prove that the Euler characteristic is additive in the following sense: for any cellular space X and its finite cellular subspaces A and B we have

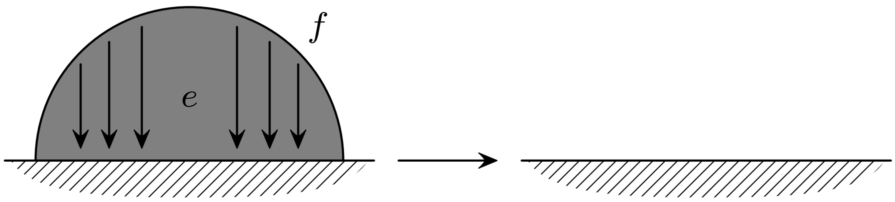
$$\chi(A \cup B) = \chi(A) + \chi(B) - \chi(A \cap B).$$

44.B. Prove that the Euler characteristic is multiplicative in the following sense: for any finite cellular spaces X and Y, the Euler characteristic of their product $X \times Y$ is $\chi(X)\chi(Y)$.

⌈44′2⌋ Collapse and Generalized Collapse

Let X be a cellular space, e and f its open cells of dimensions n and $n-1$, respectively. Suppose:

- the attaching map $\varphi_e : S^{n-1} \to X_{n-1}$ of e determines a homeomorphism of the open upper hemisphere S^{n-1}_+ onto f,
- f does not meet the images of attaching maps of cells distinct from e,
- the cell e is disjoint from the image of the attaching map of any cell.



44.C. $X \smallsetminus (e \cup f)$ is a cellular subspace of X.

44.D. *$X \smallsetminus (e \cup f)$ is a deformation retract of X.*

We say that $X \smallsetminus (e \cup f)$ is obtained from X by an ***elementary collapse***, and we write $X \searrow X \smallsetminus (e \cup f)$.

If a cellular subspace A of a cellular space X is obtained from X by a sequence of elementary collapses, then we say that X ***is collapsed*** onto A and also write $X \searrow A$.

44.E. *Collapsing does not change the Euler characteristic: if X is a finite cellular space and $X \searrow A$, then $\chi(A) = \chi(X)$.*

As above, let X be a cellular space, let e and f be its open cells of dimensions n and $n-1$, respectively, and let the attaching map $\varphi_e : S^n \to X_{n-1}$ of e determine a homeomorphism of S^{n-1}_+ onto f. Unlike the preceding situation, here we assume neither that f is disjoint from the images of attaching maps of cells different from e, nor that e is disjoint from the images of attaching maps of whatever cells. Let $\chi_e : D^n \to X$ be a characteristic map of e. Furthermore, let $\psi : D^n \to S^{n-1} \smallsetminus \varphi_e^{-1}(f) = S^{n-1} \smallsetminus S^{n-1}_+$ be a deformation retraction.

44.F. Under these conditions, the quotient space $X/_{[\chi_e(x) \sim \varphi_e(\psi(x))]}$ of X is a cellular space where the cells are the images under the natural projections of all cells of X except e and f.

We say that the cellular space $X/_{[\chi_e(x) \sim \varphi_e(\psi(x))]}$ is obtained by *cancellation of cells* e and f.

44.G. *The projection $X \to X/_{[\chi_e(x) \sim \varphi_e(\psi(x))]}$ is a homotopy equivalence.*

44.G.1. Find a cellular subspace Y of a cellular space X such that the projection $Y \to Y/_{[\chi_e(x) \sim \varphi_e(\psi(x))]}$ would be a homotopy equivalence by Theorem *44.D*.

44.G.2. Extend the map $Y \to Y \smallsetminus (e \cup f)$ to a map $X \to X'$, which is a homotopy equivalence by *44.6x*.

⌈44′3x⌋ Homotopy Equivalences of Cellular Spaces

44.1x. Let $X = A \cup_\varphi D^n$ be the space obtained by attaching an n-disk to a topological space A via a continuous map $\varphi : S^{n-1} \to A$. Prove that the complement $X \smallsetminus x$ of any point $x \in X \smallsetminus A$ admits a (strong) deformation retraction to A.

44.2x. Let X be an n-dimensional cellular space, and let K be a set intersecting each of the open n-cells of X at a single point. Prove that the $(n-1)$-skeleton X_{n-1} of X is a deformation retract of $X \smallsetminus K$.

44.3x. Prove that the complement $\mathbb{R}P^n \smallsetminus$ point is homotopy equivalent to $\mathbb{R}P^{n-1}$; the complement $\mathbb{C}P^n \smallsetminus$ point is homotopy equivalent to $\mathbb{C}P^{n-1}$.

44.4x. Prove that the punctured solid torus $D^2 \times S^1 \smallsetminus$ point, where point is an arbitrary interior point, is homotopy equivalent to a torus with a disk attached along the meridian $S^1 \times 1$.

44.5x. Let A be cellular space of dimension n, and let $\varphi : S^n \to A$ and $\psi : S^n \to A$ be two continuous maps. Prove that if φ and ψ are homotopic, then the spaces $X_\varphi = A \cup_\varphi D^{n+1}$ and $X_\psi = A \cup_\psi D^{n+1}$ are homotopy equivalent.

Below we need a more general fact.

44.6x. Let $f : X \to Y$ be a homotopy equivalence, and let $\varphi : S^{n-1} \to X$ and $\varphi' : S^{n-1} \to Y$ continuous maps. Prove that if $f \circ \varphi \sim \varphi'$, then $X \cup_\varphi D^n \simeq Y \cup_{\varphi'} D^n$.

44.7x. Let X be a space obtained from a circle by attaching two copies of a disk by the maps $S^1 \to S^1 : z \mapsto z^2$ and $S^1 \to S^1 : z \mapsto z^3$, respectively. Find a cellular space homotopy equivalent to X with the smallest possible number of cells.

44.8x. *Riddle.* Generalize the result of Problem *44.7x*.

44.9x. Prove that the space K obtained by attaching a disk to the torus $S^1 \times S^1$ along the fibre $S^1 \times 1$ is homotopy equivalent to the bouquet $S^2 \vee S^1$.

44.10x. Prove that the torus $S^1 \times S^1$ with two disks attached along the meridian $\{1\} \times S^1$ and parallel $S^1 \times 1$, respectively, is homotopy equivalent to S^2.

44.11x. Consider three circles in $\mathbb{R}^3$: $S_1 = \{x^2 + y^2 = 1,\ z = 0\}$, $S_2 = \{x^2 + y^2 = 1,\ z = 1\}$, and $S_3 = \{z^2 + (y-1)^2 = 1,\ x = 0\}$. Since $\mathbb{R}^3 \cong S^3 \smallsetminus$ point, we can assume that S_1, S_2, and S_3 lie in S^3. Prove that the space $X = S^3 \smallsetminus (S_1 \cup S_2)$ is not homotopy equivalent to the space $Y = S^3 \smallsetminus (S_1 \cup S_3)$.

44.Hx. Let X be a cellular space, $A \subset X$ a cellular subspace. Then the union $(X \times 0) \cup (A \times I)$ is a retract of the cylinder $X \times I$.

44.Ix. Let X be a cellular space, $A \subset X$ a cellular subspace. Assume that we are given a map $F : X \to Y$ and a homotopy $h : A \times I \to Y$ of the restriction $f = F|_A$. Then the homotopy h extends to a homotopy $H : X \times I \to Y$ of F.

44.Jx. Let X be a cellular space, $A \subset X$ a contractible cellular subspace. Then the projection $\mathrm{pr} : X \to X/_A$ is a homotopy equivalence.

Problem *44.Jx* implies the following assertions.

44.Kx. If a cellular space X contains a closed 1-cell e homeomorphic to I, then X is homotopy equivalent to the cellular space $X/_e$ obtained by contraction of e.

44.Lx. *Any connected cellular space is homotopy equivalent to a cellular space with one-point* 0*-skeleton.*

44.Mx. *A simply connected finite* 2*-dimensional cellular space is homotopy equivalent to a cellular space with one-point* 1*-skeleton.*

44.12x. Solve Problem *44.9x* with the help of Theorem *44.Jx*.

44.13x. Prove that the quotient space

$$\mathbb{C}P^2/_{[(z_0 : z_1 : z_2) \sim (\overline{z_0} : \overline{z_1} : \overline{z_2})]}$$

of the complex projective plane $\mathbb{C}P^2$ is homotopy equivalent to S^4.

Information. We have $\mathbb{C}P^2/_{[z \sim \tau(z)]} \cong S^4$.

44.Nx. Let X be a cellular space, and let A be a cellular subspace of X such that the inclusion $\mathrm{in} : A \to X$ is a homotopy equivalence. Then A is a deformation retract of X.

45. One-Dimensional Cellular Spaces

⌈45′1⌋ Homotopy Classification

45.A. *Any connected finite 1-dimensional cellular space is homotopy equivalent to a bouquet of circles.*

45.A.1 Lemma. Let X be a 1-dimensional cellular space, and let e be a 1-cell of X attached by an injective map $S^0 \to X_0$ (i.e., e has two distinct endpoints). Prove that the projection $X \to X/_e$ is a homotopy equivalence. Describe the homotopy inverse map explicitly.

45.B. *A finite connected cellular space X of dimension one is homotopy equivalent to the bouquet of $1-\chi(X)$ circles, and its fundamental group is a free group of rank $1-\chi(X)$.*

45.C Corollary. *The Euler characteristic of a finite connected one-dimensional cellular space is invariant under homotopy equivalence. It is not greater than one. It equals one iff the space is homotopy equivalent to point.*

45.D Corollary. *The Euler characteristic of a finite one-dimensional cellular space is not greater than the number of its connected components. It is equal to this number iff each of its connected components is homotopy equivalent to a point.*

45.E Homotopy Classification of Finite 1-Dimensional Cellular Spaces. *Finite connected one-dimensional cellular spaces are homotopy equivalent, iff their fundamental groups are isomorphic, iff their Euler characteristics are equal.*

45.1. The fundamental group of a 2-sphere punctured at n points is a free group of rank $n-1$.

45.2. Prove that the Euler characteristic of a cellular space homeomorphic to S^2 is equal to 2.

45.3 The Euler Theorem. For any convex polyhedron in $\mathbb{R}^3$, the sum of the number of its vertices and the number of its faces equals the number of its edges plus two.

45.4. Prove the Euler Theorem without using fundamental groups.

45.5. Prove that the Euler characteristic of any cellular space homeomorphic to the torus is equal to 0.

Information. The Euler characteristic is homotopy invariant, but the usual proof of this fact involves the machinery of singular homology theory, which lies far beyond the scope of our book.

⌈45′2⌋ Spanning Trees

A one-dimensional cellular space is a ***tree*** if it is connected, while the complement of each of its (open) 1-cells is disconnected. A cellular subspace A of a cellular space X is a ***spanning tree*** of X if A is a tree and is not contained in any other cellular subspace $B \subset X$ which is a tree.

45.F. Any finite connected one-dimensional cellular space contains a spanning tree.

45.G. Prove that a cellular subspace A of a cellular space X is a spanning tree iff A is a tree and contains all vertices of X.

Theorem *45.G* explains the term *spanning tree*.

45.H. Prove that a cellular subspace A of a cellular space X is a spanning tree iff it is a tree and the quotient space $X/_A$ is a bouquet of circles.

45.I. Let X be a one-dimensional cellular space, A its cellular subspace. Prove that if A is a tree, then the projection $X \to X/_A$ is a homotopy equivalence.

Problems *45.F*, *45.I*, and *45.H* provide one more proof of Theorem *45.A*.

⌈45′3x⌋ Dividing Cells

45.Jx. *In a one-dimensional connected cellular space, each connected component of the complement of an edge meets the closure of the edge. The complement has at most two connected components.*

A complete local characterization of a vertex in a one-dimensional cellular space is its ***degree***. This is the total number of points in the preimages of the vertex under attaching maps of all one-cells of the space. It is more traditional to define the degree of a vertex v as the number of edges incident to v, counting with multiplicity 2 the edges that are incident only to v.

45.Kx. *1) Each connected component of the complement of a vertex in a connected one-dimensional cellular space contains an edge with boundary containing the vertex. 2) The complement of a vertex of degree m has at most m connected components.*

⌈45′4x⌋ Trees and Forests

A one-dimensional cellular space is a ***tree*** if it is connected, while the complement of each of its (open) 1-cells is disconnected. A one-dimensional cellular space is a ***forest*** if each of its connected components is a tree.

45.Lx. Any cellular subspace of a forest is a forest. In particular, any connected cellular subspace of a tree is a tree.

45.Mx. In a tree, the complement of an edge has two connected components.

45.Nx. In a tree, the complement of a vertex of degree m has m connected components.

45.Ox. A finite tree has a vertex of degree one.

45.Px. Any finite tree collapses to a point and has Euler characteristic one.

45.Qx. Prove that any point of a tree is its deformation retract.

45.Rx. Any finite one-dimensional cellular space that can be collapsed to a point is a tree.

45.Sx. In any finite one-dimensional cellular space, the sum of degrees of all vertices is twice the number of edges.

45.Tx. A finite connected one-dimensional cellular space with Euler characteristic one has a vertex of degree one.

45.Ux. A finite connected one-dimensional cellular space with Euler characteristic one collapses to a point.

⌈45′5x⌋ Simple Paths

Let X be a one-dimensional cellular space. A *simple path of length* n in X is a finite sequence $(v_1, e_1, v_2, e_2, \dots, e_n, v_{n+1})$ formed by vertices v_i and edges e_i of X such that each term appears in it only once and the boundary of every edge e_i consists of the preceding and subsequent vertices v_i and v_{i+1}. The vertex v_1 is the *initial* vertex, and v_{n+1} is the *final* one. The simple path *connects* these vertices. They are connected by a path $I \to X$, which is a topological embedding with image contained in the union of all cells involved in the simple path. The union of these cells is a cellular subspace of X. It is called a *simple broken line*.

45.Vx. *In a connected one-dimensional cellular space, any two vertices are connected by a simple path.*

45.Wx Corollary. *In a connected one-dimensional cellular space X, any two points are connected by a path $I \to X$ which is a topological embedding.*

> ***45.6x.*** Can a path-connected space contain two distinct points that cannot be connected by a path which is a topological embedding?
>
> ***45.7x.*** Can you find a Hausdorff space with this property?

45.Xx. *A connected one-dimensional cellular space X is a tree iff there exists no topological embedding $S^1 \to X$.*

45.Yx. *In a one-dimensional cellular space X, there exists a non-null-homotopic loop $S^1 \to X$ iff there exists a topological embedding $S^1 \to X$.*

45.Zx. *A one-dimensional cellular space is a tree iff any two distinct vertices are connected in it by a unique simple path.*

45.8x. Prove that any finite tree has fixed point property.

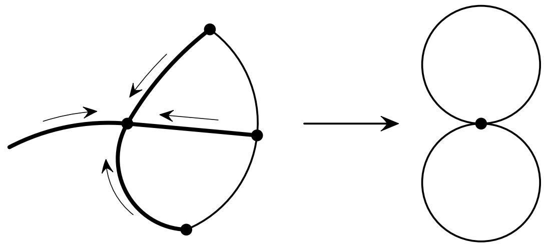
Cf. *38.12*, *38.13*, and *38.14*.

45.9x. Is this true for each tree? For each finite connected one-dimensional cellular space?

46. Fundamental Group of a Cellular Space

⌈46′1⌋ One-Dimensional Cellular Spaces

46.A. *The fundamental group of a connected finite one-dimensional cellular space X is a free group of rank $1 - \chi(X)$.*



46.B. Let X be a finite connected one-dimensional cellular space, T a spanning tree of X, and $x_0 \in T$. For each 1-cell $e \subset X \smallsetminus T$, choose a loop s_e that starts at x_0, goes inside T to e, then goes once along e, and then returns to x_0 in T. Prove that $\pi_1(X, x_0)$ is freely generated by the homotopy classes of s_e.

⌈46′2⌋ Generators

46.C. Let A be a topological space, $x_0 \in A$. Let $\varphi : S^{k-1} \to A$ be a continuous map, $X = A \cup_\varphi D^k$. If $k > 1$, then the inclusion homomorphism $\pi_1(A, x_0) \to \pi_1(X, x_0)$ is surjective. Cf. *46.G.4* and *46.G.5*.

46.D. *Let X be a cellular space, let x_0 be its* 0*-cell, and let X_1 be the* 1*-skeleton of X. Then the inclusion homomorphism*

$$\pi_1(X_1, x_0) \to \pi_1(X, x_0)$$

is surjective.

46.E. Let X be a finite cellular space, T a spanning tree of X_1, and $x_0 \in T$. For each cell $e \subset X_1 \smallsetminus T$, choose a loop s_e that starts at x_0, goes inside T to e, then goes once along e, and finally returns to x_0 in T. Prove that $\pi_1(X, x_0)$ is generated by the homotopy classes of s_e.

46.1. Deduce Theorem *32.G* from Theorem *46.D*.

46.2. Find $\pi_1(\mathbb{C}P^n)$.

⌈46′3⌋ Relations

Let X be a cellular space, x_0 its 0-cell. Denote by X_n the n-skeleton of X. Recall that X_2 is obtained from X_1 by attaching copies of the disk

D^2 via continuous maps $\varphi_\alpha : S^1 \to X_1$. The attaching maps are circular loops in X_1. For each α, choose a path $s_\alpha : I \to X_1$ connecting $\varphi_\alpha(1)$ with x_0. Denote by N the normal subgroup of $\pi_1(X, x_0)$ generated (as a normal subgroup[4]) by the elements

$$T_{s_\alpha}[\varphi_\alpha] \in \pi_1(X_1, x_0).$$

46.F. N does not depend on the choice of the paths s_α.

46.G. *The normal subgroup N is the kernel of the inclusion homomorphism* $\mathrm{in}_* : \pi_1(X_1, x_0) \to \pi_1(X, x_0)$.

Theorem *46.G* can be proved in various ways. For example, we can derive it from the Seifert–van Kampen Theorem (see *46.7x*). Here we prove Theorem *46.G* by constructing a "rightful" covering space. The inclusion $N \subset \operatorname{Ker} \mathrm{in}_*$ is rather obvious (see *46.G.1*). The proof of the converse inclusion involves the existence of a covering $p : Y \to X$ whose submap over the 1-skeleton of X is a covering $p_1 : Y_1 \to X_1$ with group N, and the fact that $\operatorname{Ker} \mathrm{in}_*$ is contained in the group of each covering over X_1 that extends to a covering over the entire X. The scheme of the argument suggested in Lemmas 1–7 can also be modified. The thing is that the inclusion $X_2 \to X$ induces an isomorphism of fundamental groups. It is not difficult to prove this, but the techniques involved, though quite general and natural, nevertheless lie beyond the scope of our book. Here we just want to emphasize that this result replaces Lemmas 4 and 5.

46.G.1 Lemma 1. $N \subset \operatorname{Ker} i_*$, cf. *32.J* (3).

46.G.2 Lemma 2. Let $p_1 : Y_1 \to X_1$ be a covering with covering group N. Then for any α and any point $y \in p_1^{-1}(\varphi_\alpha(1))$ the loop φ_α has a lift $\widetilde{\varphi}_\alpha : S^1 \to Y_1$ with $\widetilde{\varphi}_\alpha(1) = y$.

46.G.3 Lemma 3. Let Y_2 be a cellular space obtained by attaching copies of a disk to Y_1 along all lifts of attaching maps φ_α. Then there exists a map $p_2 : Y_2 \to X_2$ that extends p_1 and is a covering.

46.G.4 Lemma 4. Attaching maps of n-cells with $n \geq 3$ lift to any covering space. Cf. *41.Xx* and *41.Yx*.

46.G.5 Lemma 5. Covering $p_2 : Y_2 \to X_2$ extends to a covering of the whole X.

46.G.6 Lemma 6. Any loop $s : I \to X_1$ realizing an element of $\operatorname{Ker} i_*$ (i.e., null-homotopic in X) is covered by a loop of Y. The covering loop is contained in Y_1.

46.G.7 Lemma 7. $N = \operatorname{Ker} \mathrm{in}_*$.

[4]Recall that a subgroup N is *normal* if N coincides with all conjugate subgroups of N. The normal subgroup N generated by a set A is the minimal normal subgroup containing A. As a subgroup, N is generated by elements of A and elements conjugate to them. This means that each element of N is a product of elements conjugate to elements of A.

46.H. *The inclusion* $\text{in}_2 : X_2 \to X$ *induces an isomorphism between the fundamental groups of a cellular space and its* 2*-skeleton.*

46.3. Check that the covering over the cellular space X constructed in the proof of Theorem *46.G* is universal.

⌈46′4⌋ Writing Down Generators and Relations

Theorems *46.E* and *46.G* imply the following recipe for writing down a presentation for the fundamental group of a finite dimensional cellular space by generators and relations:

Let X be a finite cellular space, x_0 a 0-cell of X. Let T be a spanning tree of the 1-skeleton of X. For each 1-cell $e \not\subset T$ of X, choose a loop s_e that starts at x_0, goes inside T to e, goes once along e, and then returns to x_0 in T. Let $g_1, \dots, g_m$ be the homotopy classes of these loops. Let $\varphi_1, \dots, \varphi_n : S^1 \to X_1$ be the attaching maps of 2-cells of X. For each φ_i, choose a path s_i connecting $\varphi_i(1)$ with x_0 in the 1-skeleton of X. Express the homotopy class of the loop $s_i^{-1}\varphi_i s_i$ as a product of powers of generators g_j. Let $r_1, \dots, r_n$ be the words in letters $g_1, \dots, g_m$ obtained in this way. The fundamental group of X is generated by $g_1, \dots, g_m$, which satisfy the defining relations $r_1 = 1, \dots, r_n = 1$.

46.I. Check that this rule gives correct answers in the cases of $\mathbb{R}P^n$ and $S^1 \times S^1$ for the cellular presentations of these spaces provided in Problems *42.H* and *42.E*.

In assertion *44.Mx* proved above, we assumed that the cellular space is 2-dimensional. The reason for this was that at that moment we did not know that the inclusion $X_2 \to X$ induces an isomorphism of fundamental groups.

46.J. Each finite simply connected cellular space is homotopy equivalent to a cellular space with one-point 1-skeleton.

⌈46′5⌋ Fundamental Groups of Basic Surfaces

46.K. *The fundamental group of a sphere with g handles admits the following presentation:*

$$\langle a_1, b_1, a_2, b_2, \dots a_g, b_g \mid a_1 b_1 a_1^{-1} b_1^{-1} a_2 b_2 a_2^{-1} b_2^{-1} \dots a_g b_g a_g^{-1} b_g^{-1} = 1 \rangle.$$

46.L. *The fundamental group of a sphere with g cross-caps admits the following presentation:*

$$\langle a_1, a_2, \dots a_g \mid a_1^2 a_2^2 \dots a_g^2 = 1 \rangle.$$

46.M. *Spheres with different numbers of handles have non-isomorphic fundamental groups.*

When we want to prove that two finitely presented groups are not isomorphic, one of the first natural moves is to abelianize the groups. (Recall that to *abelianize* a group G means to quotient G out by the commutator subgroup. The commutator subgroup $[G,G]$ is the normal subgroup generated by the commutators $a^{-1}b^{-1}ab$ for all $a, b \in G$. Abelianization means adding relations $ab = ba$ for any $a, b \in G$.)

Abelian finitely generated groups are well known. Any finitely generated Abelian group is isomorphic to a product of a finite number of cyclic groups. If the abelianized groups are not isomorphic, then the original groups are not isomorphic as well.

46.M.1. *The abelianized fundamental group of a sphere with g handles is a free Abelian group of rank $2g$ (i.e., is isomorphic to $\mathbb{Z}^{2g}$).*

46.N. *Fundamental groups of spheres with different numbers of cross-caps are not isomorphic.*

46.N.1. *The abelianized fundamental group of a sphere with g cross-caps is isomorphic to $\mathbb{Z}^{g-1} \times \mathbb{Z}_2$.*

46.O. *Spheres with different numbers of handles are not homotopy equivalent.*

46.P. *Spheres with different numbers of cross-caps are not homotopy equivalent.*

46.Q. *A sphere with handles is not homotopy equivalent to a sphere with cross-caps.*

If X is a path-connected space, then the abelianized fundamental group of X is the 1-*dimensional* (or *first*) *homology group* of X and denoted by $H_1(X)$. If X is not path-connected, then $H_1(X)$ is the direct sum of the first homology groups of all path-connected components of X. Thus *46.M.1* can be rephrased as follows: if F_g is a sphere with g handles, then $H_1(F_g) = \mathbb{Z}^{2g}$.

⌈46′6x⌋ Seifert–van Kampen Theorem

To calculate fundamental group, one often uses the Seifert–van Kampen Theorem, instead of the cellular techniques presented above.

46.Rx Seifert–van Kampen Theorem. *Let X be a path-connected topological space, let A and B be its open path-connected subspaces covering X, and let $C = A \cap B$ be also path-connected. Then $\pi_1(X)$ can be presented as the amalgamated product of $\pi_1(A)$ and $\pi_1(B)$ with identified subgroup $\pi_1(C)$. In other words, if $x_0 \in C$,*

$$\pi_1(A, x_0) = \langle \alpha_1, \dots, \alpha_p \mid \rho_1 = \dots = \rho_r = 1 \rangle,$$

$$\pi_1(B, x_0) = \langle \beta_1, \dots, \beta_q \mid \sigma_1 = \dots = \sigma_s = 1 \rangle,$$

$\pi_1(C, x_0)$ is generated by its elements $\gamma_1, \dots, \gamma_t$, and $in_A : C \to A$ and $in_B : C \to B$ are inclusions, then $\pi_1(X, x_0)$ can be presented as

$$\langle \alpha_1, \dots, \alpha_p, \beta_1, \dots, \beta_q \mid \rho_1 = \dots = \rho_r = \sigma_1 = \dots = \sigma_s = 1, \quad \mathrm{in}_{A*}(\gamma_1) = \mathrm{in}_{B*}(\gamma_1), \dots, \mathrm{in}_{A*}(\gamma_t) = \mathrm{in}_{B*}(\gamma_t) \rangle.$$

Now we consider the situation where the space X and its subsets A and B are cellular.

46.Sx. Assume that X is a connected finite cellular space, and A and B are two cellular subspaces of X covering X. Denote $A \cap B$ by C. How are the fundamental groups of X, A, B, and C related to each other?

46.Tx Seifert–van Kampen Theorem. *Let X be a connected finite cellular space, let A and B be two connected cellular subspaces covering X, and let $C = A \cap B$. Assume that C is also connected. Let $x_0 \in C$ be a 0-cell,*

$$\pi_1(A, x_0) = \langle \alpha_1, \dots, \alpha_p \mid \rho_1 = \dots = \rho_r = 1 \rangle,$$
$$\pi_1(B, x_0) = \langle \beta_1, \dots, \beta_q \mid \sigma_1 = \dots = \sigma_s = 1 \rangle,$$

and let the group $\pi_1(C, x_0)$ be generated by the elements $\gamma_1, \dots, \gamma_t$. Denote by $\xi_i(\alpha_1, \dots, \alpha_p)$ and $\eta_i(\beta_1, \dots, \beta_q)$ the images of the elements γ_i (more precisely, their expression via the generators) under the inclusion homomorphisms

$$\pi_1(C, x_0) \to \pi_1(A, x_0) \text{ and, respectively, } \pi_1(C, x_0) \to \pi_1(B, x_0).$$

Then

$$\pi_1(X, x_0) = \langle \alpha_1, \dots, \alpha_p, \beta_1, \dots, \beta_q \mid \rho_1 = \dots = \rho_r = \sigma_1 = \dots = \sigma_s = 1, \quad \xi_1 = \eta_1, \dots, \xi_t = \eta_t \rangle.$$

46.4x. Let X, A, B, and C be as above. Assume that A and B are simply connected and C has two connected components. Prove that $\pi_1(X)$ is isomorphic to $\mathbb{Z}$.

46.5x. Is Theorem *46.Tx* a special case of Theorem *46.Rx*?

46.6x. May the assumption of openness of A and B in *46.Rx* be omitted?

46.7x. Deduce Theorem *46.G* from the Seifert–van Kampen Theorem *46.Rx*.

46.8x. Compute the fundamental group of the *lens space*, which is obtained by pasting together two solid tori via the homeomorphism $S^1 \times S^1 \to S^1 \times S^1 : (u, v) \mapsto (u^k v^l, u^m v^n)$, where $kn - lm = 1$.

46.9x. Determine the homotopy and the topological type of the lens space for $m = 0, 1$.

46.10x. Find a presentation for the fundamental group of the complement in $\mathbb{R}^3$ of a torus knot K of type (p,q), where p and q are relatively prime positive integers. This knot lies on the revolution torus T, which is described by parametric equations

$$\begin{cases} x = & (2+\cos 2\pi u)\cos 2\pi v \\ y = & (2+\cos 2\pi u)\sin 2\pi v \\ z = & \sin 2\pi u, \end{cases}$$

and K is described on T by equation $pu = qv$.

46.11x. Let (X, x_0) and (Y, y_0) be two simply connected topological spaces with marked points, and let $Z = X \vee Y$ be their bouquet.

(1) Prove that if X and Y are cellular spaces, then Z is simply connected.
(2) Prove that if x_0 and y_0 have neighborhoods $U_{x_0} \subset X$ and $V_{y_0} \subset Y$ that admit strong deformation retractions to x_0 and y_0, respectively, then Z is simply connected.
(3) Construct two simply connected topological spaces X and Y with a non-simply connected bouquet.

⌈46′7x⌋ Group-Theoretic Digression: Amalgamated Product of Groups

At first glance, description of the fundamental group of X given above in the statement of the Seifert–van Kampen Theorem is far from being invariant: it depends on the choice of generators and relations of other groups involved. However, this is actually a detailed description of a group-theoretic construction in terms of generators and relations. After solving the next problem, you will get a more complete picture of the subject.

46.Ux. Let A and B be two groups:

$$A = \langle \alpha_1, \dots, \alpha_p \mid \rho_1 = \dots = \rho_r = 1 \rangle,$$

$$B = \langle \beta_1, \dots, \beta_q \mid \sigma_1 = \dots = \sigma_s = 1 \rangle,$$

and let C be a group generated by $\gamma_1, \dots \gamma_t$. Let $\xi : C \to A$ and $\eta : C \to B$ be arbitrary homomorphisms. Then

$$\begin{aligned} X = \langle \alpha_1, \dots, \alpha_p, \beta_1, \dots, \beta_q \mid \\ \rho_1 = \dots = \rho_r = \sigma_1 = \dots = \sigma_s = 1, \\ \xi(\gamma_1) = \eta(\gamma_1), \dots, \xi(\gamma_t) = \eta(\gamma_t) \rangle, \end{aligned}$$

and homomorphisms $\phi : A \to X : \alpha_i \mapsto \alpha_i,\ i = 1, \dots, p$ and $\psi : B \to X : \beta_j \mapsto \beta_j,\ j = 1, \dots, q$ take part in commutative diagram

$$\begin{tikzcd} & A \arrow[dr, "\xi"] & \\ C \arrow[ur, "\phi"] \arrow[dr, "\psi"'] & & X \\ & B \arrow[ur, "r"'] & \end{tikzcd}$$

and for each group X' and homomorphisms $\varphi' : A \to X'$ and $\psi' : B \to X'$ involved in commutative diagram

$$\begin{tikzcd} & A \arrow[dr, "\xi'"] & \\ C \arrow[ur, "\phi"] \arrow[dr, "\psi"'] & & X' \\ & B \arrow[ur, "r'"'] & \end{tikzcd}$$

there exists a unique homomorphism $\zeta : X \to X'$ such that diagram

$$\begin{tikzcd} & A \arrow[dr, "\xi"'] \arrow[drr, "\xi'"] & & \\ C \arrow[ur, "\phi"] \arrow[dr, "\psi"'] & & X \arrow[r, dashed, "\zeta"] & X' \\ & B \arrow[ur, "r"] \arrow[urr, "r'"'] & & \end{tikzcd}$$

is commutative. The latter determines the group X up to isomorphism.

The group X described in *46.Ux* is a *free product of A and B with amalgamated subgroup C*. It is denoted by $A *_C B$. Notice that the name is not quite precise, since it ignores the role of the homomorphisms ϕ and ψ and the possibility that they may be not injective.

If the group C is trivial, then $A *_C B$ is denoted by $A * B$ and called the *free product* of A and B.

46.12x. Is a free group of rank n a free product of n copies of $\mathbb{Z}$?

46.13x. Represent the fundamental group of Klein bottle as $\mathbb{Z} *_{\mathbb{Z}} \mathbb{Z}$. Does this decomposition correspond to a decomposition of Klein bottle?

46.14x. Riddle. Define a free product as a set of equivalence classes of words in which the letters are elements of the factors.

46.15x. Investigate algebraic properties of free multiplication of groups: is it associative, commutative and, if it is, then in what sense? Do homomorphisms of the factors determine a homomorphism of the product?

46.16x*. Find decomposition of the modular group

$$Mod = SL(2,\mathbb{Z})/\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

as a free product $\mathbb{Z}_2 * \mathbb{Z}_3$.

⌈46′8x⌋ Addendum to Seifert–van Kampen Theorem

The Seifert–van Kampen Theorem appeared and is used mostly as a tool for calculation of fundamental groups. However, it does not help in many situations. For example, it does not work under the assumptions of the following theorem.

46.Vx. *Let X be a topological space, A and B open sets covering X, and $C = A \cap B$. Assume that A and B are simply connected and C has two connected components. Then $\pi_1(X)$ is isomorphic to $\mathbb{Z}$.*

Theorem *46.Vx* also holds true if we assume that C has two path-connected components. The difference seems to be immaterial, but the proof becomes incomparably more technical.

Seifert and van Kampen needed a more universal tool for calculation of fundamental groups, and theorems they published were much more general than Theorem *46.Rx*. Theorem *46.Rx* is all that could find its way from the original papers to textbooks. Theorem *46.4x* is another special case of their results. The most general formulation is rather cumbersome, and we restrict ourselves to one more special case that was distinguished by van Kampen. Together with *46.Rx*, it allows one to calculate fundamental groups in all situations that are available with the most general formulations by van Kampen, although not that fast. We formulate the original version of this theorem, but first we recommend starting with a cellular version, in which the results presented in the beginning of this section allow one to obtain a complete answer about calculation of fundamental groups. After that is done consider the general situation.

First, let us describe the situation common for both formulations. Let A be a topological space, B its closed subset, and U a neighborhood of B in A such that $U \setminus B$ is the union of two disjoint sets, M_1 and M_2, open in A. Put $N_i = B \cup M_i$. Let C be a topological space that can be represented as $(A \setminus U) \cup (N_1 \sqcup N_2)$ and such that the sets $(A \setminus U) \cup N_1$ and $(A \setminus U) \cup N_2$ with the topology induced from A form a fundamental cover of C. There are two copies of B in C, which come from N_1 and N_2. The space A can be identified with the quotient space of C obtained by identifying the two copies of B via the natural homeomorphism. However, our description begins with A, since this is the space whose fundamental group we want to calculate, while the space B is auxiliary constructed out of A (see Figure 1).

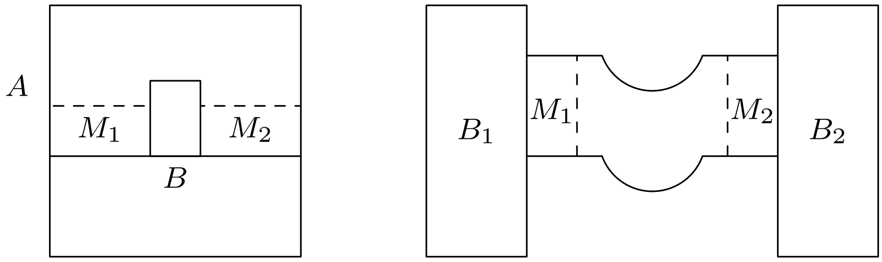


Figure 1

In the cellular version of the statement formulated below, it is supposed that the space A is cellular and B is its cellular subspace. Then C is also equipped with a natural cellular structure such that the natural map $C \to A$ is cellular.

46.Wx. *In the situation described above, assume that C is path-connected and $x_0 \in C \smallsetminus (B_1 \cup B_2)$. Let $\pi_1(C, x_0)$ be presented by generators $\alpha_1, \dots, \alpha_n$ and relations $\psi_1 = 1, \dots, \psi_m = 1$. Assume that base points $y_i \in B_i$ are mapped to the same point y under the map $C \to A$, and σ_i is a homotopy class of a path connecting x_0 with y_i in C. Let $\beta_1, \dots, \beta_p$ be generators of $\pi_1(B, y)$, and let $\beta_{1i}, \dots, \beta_{pi}$ be the corresponding elements of $\pi_1(B_i, y_i)$. Denote by φ_{li} a word representing $\sigma_i \beta_{li} \sigma_i^{-1}$ in terms of $\alpha_1, \dots, \alpha_n$. Then $\pi_1(A, x_0)$ has the following presentation:*

$$\langle \alpha_1, \dots, \alpha_n, \gamma \mid \psi_1 = \dots = \psi_m = 1, \gamma\varphi_{11} = \varphi_{12}\gamma, \dots, \gamma\varphi_{p1} = \varphi_{p2}\gamma \rangle.$$

46.17x. Using *46.Wx*, calculate the fundamental groups of the torus and the Klein bottle.

46.18x. Using *46.Wx*, calculate the fundamental groups of basic surfaces.

46.19x. Deduce Theorem *46.4x* from *46.Rx* and *46.Wx*.

46.20x. Riddle. Develop an algebraic theory of the group-theoretic construction contained in Theorem *46.Wx*.

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