(Co) limits, monos & epis.

Structure of the course.

B
- Categories.
- (Co) limits.
- Adjunctions

Core
- Summing up: Poset. + prequel of Yoneda
- Yoneda lemma.
- Colimits II

Adv
- Intro via Posets
- Monads ⟶ Closure operators
- Kan extensions ⟶ solutions to extension problem.
- Monoidal cats & enrichments

Last time?

Categories
functor
⟶ nat transformation

new ideas

→ a monoid is a category.
→ a poset is a category.
→ there is a category Cat of small categories.
→ opposite of a category $\mathcal{C}^{op}$ $\mathcal{C}^{o}$

$\mathbb{P} \not\cong \mathbb{P}^{op}$

$\to \mathcal{C}, \mathcal{D}$, $\mathrm{Cat}(\mathcal{C}, \mathcal{D})$

functors nat trans.

Impo ex

Quiv $\supseteq$ $\mathbf{2}$ Set $\rightrightarrows$

eg of cats

functor cats.

nat trans

(Co) limits , epis & mono.

Products,
Equalizers, $\Longrightarrow$ limit
Pull backs.

Product

$$\begin{tikzcd}
 & S \arrow[dl, bend right, "d"'] \arrow[d, dotted, "\exists!"] \arrow[dr, bend left, "t"] & \\
b & a \times b \arrow[l, "\pi_b"] \arrow[r, "\pi_a"'] & a
\end{tikzcd}$$

$\forall d, t.$

$\mathcal{C}$ category
$a, b \in \mathcal{C}$

Rem the product of a, b is $(a \times b, \pi_a, \pi_b)$.

Ex In Set, a, b

$$a \times b = \{(a, b) : a \in A, b \in B\}.$$

this is more or less an axiom of ZFC

$$\begin{tikzcd} & a \times b \arrow[dl, "\pi_b"'] \arrow[dr, "\pi_a"] & \\ b & & a \end{tikzcd}$$

Ex In spaces Top.

$$\begin{tikzcd} & (a \times b, \tau) \arrow[dl, "\pi_b"'] \arrow[dr, "\pi_a"] & \\ b & & a \end{tikzcd}$$

τ = smallest topology making the projections continuous.

Ex Vect

V, W

$$\begin{tikzcd} & V \oplus W \arrow[dl, "\pi_a"'] \arrow[dr, "\pi_b"] & \\ v & (v, w) & w. \end{tikzcd}$$

Ex $(\mathbb{P}, \leqslant)$

$$\begin{tikzcd} & d \arrow[d, "\wedge"] \arrow[ddl, bend right, "\pi"'] \arrow[ddr, bend left, "\pi"] & \\ & a \times b \arrow[dl, "\geq"'] \arrow[dr, "\leq"] & \\ b & & a \end{tikzcd}$$

$a \times b =: \inf\{a, b\}$ when it exists

Def

$$\begin{tikzcd}
 & & S \arrow[d, dashed] \arrow[ddll] \arrow[ddl] \arrow[dd, bend left] \arrow[ddr] \arrow[ddrr] & & \\
 & & \prod_{i \in I} a_i \arrow[dll, "\pi_5"'] \arrow[dl, "\pi_4"'] \arrow[d, "\pi_3"] \arrow[dr, "\pi_2"] \arrow[drr, "\pi_1"] & & \\
\cdots & e_4 & e_3 & e_2 & a_1
\end{tikzcd}$$

Sanity check: $a \times b$

$$\begin{tikzcd}
 & S \arrow[ddl, "d"'] \arrow[ddr, "t"] \arrow[d, dashed, "\exists! (d,t)"] & \\
 & a \times b \arrow[dl, "\pi_a"'] \arrow[dr, "\pi_b"] & \\
a & & b
\end{tikzcd}$$

$$(d,t)(s) = (d(s), t(s)).$$

$$\pi_a (d,t)(s) = d(s)$$

Rem If a product exist it is unique up to iso.

$$\begin{tikzcd}
a \square b \arrow[r, bend left] & a \times b \arrow[l, bend left]
\end{tikzcd}$$

Rem / Ex

· Set, empty product?

Equalizers.

$$\begin{tikzcd} d \arrow[r, dashed, "\exists!"'] \arrow[rr, bend left, "\forall j"] & Eq(s,t) \arrow[r, "i_{(s,t)}"] & a \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & b \end{tikzcd}$$

$s \circ i_{(s,t)} = t \circ i_{(s,t)}$

$s \circ j = t \circ j$

equalizer: $(\, Eq(s,t),\ i_{(s,t)} \,)$

Ex In Set

$$\begin{tikzcd} S \arrow[r, bend left, "t"] & \{x \in a \mid f(x) = g(x)\} \arrow[r, hook, "i"] & a \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & b \end{tikzcd}$$

$x \longmapsto x$

$s \longmapsto \underbrace{t(s)}_{\in}$

In Vect

Ex

$$\begin{tikzcd} Ker(f) \arrow[r] & V \arrow[r, shift left, "f"] \arrow[r, shift right, "0"'] & W \end{tikzcd}$$

is $Eq(f, 0)$

$$\mathrm{Eq}(f,g) \qquad V \underset{g}{\overset{f}{\rightrightarrows}} W$$

$$\begin{array}{c} \mathrm{Eq}(f,g) \\ \cong \\ \mathrm{Ker}(f-g). \end{array} \qquad f-g=0. \iff f=g.$$

<u>Ex</u> Eq in Top.

$$\{x\in X : f(x)=g(x)\} \hookrightarrow X \underset{g}{\overset{f}{\rightrightarrows}} Y$$

Pull backs.

$$\begin{tikzcd}
d \arrow[dr, dashed] \arrow[drr, bend left, "e"] \arrow[ddr, bend right, "e"'] & & \\
& a\times_b c \arrow[r, "\pi_a"] \arrow[d, "\pi_c"'] & a \arrow[d, "f"] \\
& c \arrow[r, "g"'] & b
\end{tikzcd}$$

span

$$\underline{g\circ \pi_c = f\circ \pi_a}$$

<u>Ex</u> in Set

$$\begin{tikzcd}
f^{-1}(x) \arrow[r, hook, "i"] \arrow[d] & a \arrow[d, "f"] \\
\{*\} \arrow[r, "x"'] & b
\end{tikzcd}$$

Pullback are generalized fibers

Obs In a poset

$$\begin{tikzcd} a \times c \arrow[d, phantom, "\cong"] & \\ p \arrow[r, hook] \arrow[d] & a \\ c & b \arrow[u, phantom, "\geq"] \arrow[l, phantom, "\leq"] \end{tikzcd}$$

In Set

$$\begin{tikzcd} T \times G \arrow[r, phantom, "\overset{?}{\cong}"] & T \times_P G \arrow[r] \arrow[d] & T \arrow[d, "f"] \\ & G \arrow[r, "s"'] & P \end{tikzcd}$$

No!

For which f, s

$$T \times G \cong \begin{tikzcd} & G \arrow[d, "f"] \\ T \arrow[r, "s"'] & ? \end{tikzcd}$$

Product	Eq	Pullback.
	$\{e : f(e) = g(e)\}$	$\{(a,c) : f(a) = g(c)\}$

$$\begin{tikzcd} \{(a,c) : f(a) = g(c)\} \arrow[r, hook, "\pi_A"] \arrow[d, "\pi_c"'] & A \arrow[d, "f"] \\ C \arrow[r, "p"'] & B \end{tikzcd}$$

Rem In the category of Set.

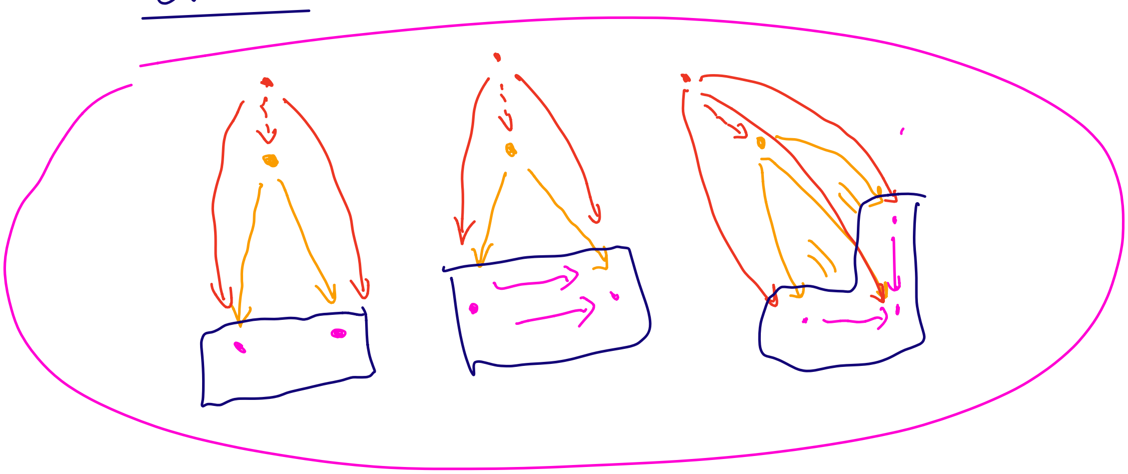
$$\boxed{A \underset{g_\alpha}{\overset{f_\alpha}{\rightrightarrows}} X_\alpha} \qquad \alpha \in I$$

$$A \underset{(g_\alpha)}{\overset{(f_\alpha)}{\rightrightarrows}} \prod_{\alpha \in I} X_\alpha$$

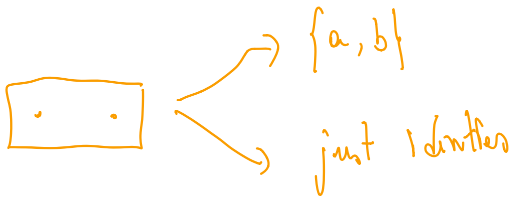
$(f_\alpha) \qquad a \longmapsto \left(f_{\alpha_1}(a), f_{\alpha_2}(a) \; \cdots \cdots \; f_{\alpha_?}(a) \right)$

$Eq\left((f_\alpha), (g_\alpha)\right)$

Limits



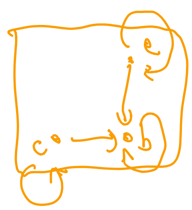
Def Let I be a small category. Then a diagram of shape I in $\mathcal{C}$ is a functor $I \longrightarrow \mathcal{C}$.



What is a functor $\boxed{\cdot\ \cdot} \xrightarrow{f} \mathcal{C}$?

$$\{\text{functors } \boxed{\cdot\ \cdot} \longrightarrow \mathcal{C}\} \cong \{\text{couples of objects in } \mathcal{C}\}.$$

Shape for pullbacks



← walking span

$$\{\text{functors } \boxed{\lrcorner} \longrightarrow \mathcal{C}\} \cong \text{spans in } \mathcal{C}.$$

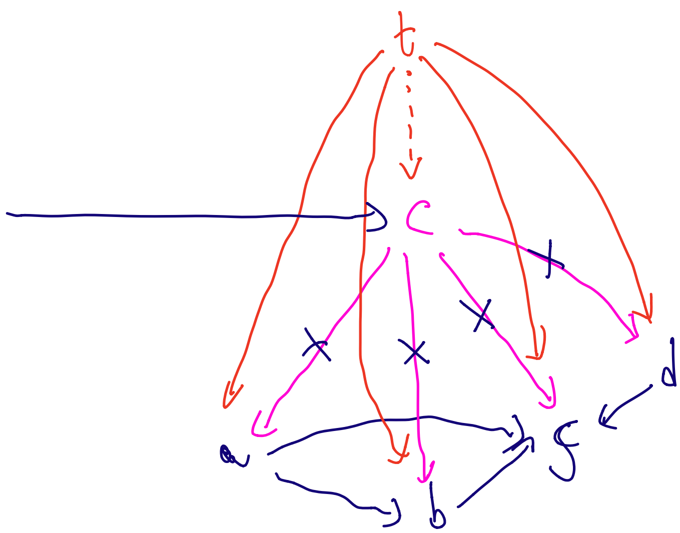
Def Given a diagram $f : I \longrightarrow \mathcal{C}$, a cone for f is the data of

$(c, \alpha_i : c \longrightarrow f(i))$ such that

$$
\begin{tikzcd}
 & c \arrow[dl, "\alpha_j"'] \arrow[dr, "\alpha_i"] \arrow[d, phantom, "="] & \\
f(j) \arrow[rr, "f(x)"] & & f(i)
\end{tikzcd}
\qquad x : j \to i \in I.
$$

<u>Def</u> a limit for a diagram $f : I \to C$ is a cone with the following universal property.

$\forall$ other cone, there exists a unique factorization



<u>Not</u> $\lim D$, $\cancel{\varprojlim D}$.

<u>Exmp</u> "inverse limits" $(\mathbb{N}\ \leq)^{op}$

$$f : \mathbb{N}^{op} \longrightarrow C$$

$$
\begin{tikzcd}
\underline{\lim f} \arrow[d] \arrow[dr] \arrow[drr] \arrow[drrr] & & & \\
\cdots \; f(4) \arrow[r] & f(3) \arrow[r] & f(2) \arrow[r] & f(1)
\end{tikzcd}
$$

Profinite group theory.

$\mathbb{Q}_p$ $\quad \mathbb{Z}/p^n \to \mathbb{Z}/p^3 \to \mathbb{Z}/p^2 \xrightarrow{\pi} \mathbb{Z}/p$

p-adics

Does the category of sets have all limits of small shape.

$$I \xrightarrow{f} \mathrm{Set}$$

I is discrete ✓ (product).

equalizers ✓

$$\lim f \cong \mathrm{Set}(1, \lim f).$$

$$\begin{array}{l} 1 \longrightarrow \lim f \\ \quad \downarrow \quad \downarrow \quad \downarrow \\ f(i) \to f(j) \to f(k) \end{array}$$

UP $\cong \{\text{cones over } f \text{ with vertex } 1\}$

$$\cong \underbrace{\left\{ (x_i)_{i \in I} : x_i \in f(i) \ \forall\, u \in I \quad f(u)(x_i) = x_j. \right\}}$$

$$\subset \prod_{i \in I} f(i)$$

it is a very huge equalizer in the product

<u>Def</u> A category is (small) complete if it has limits of (small) diagrams.

<u>thm</u> A category is complete iff it has all (small) products & equalizers.

<u>Proof</u> $\Rightarrow$ ✓ $\qquad f : \boxed{I} \longrightarrow C.$

$\Leftarrow$ $\lim f \subset \underbrace{\prod_{i \in I} D(i)}$

$$\prod_{i \in I} D(i) \underset{t}{\overset{s}{\rightrightarrows}} \prod_{u \in I(i,j)} f(j)$$

$$S : \prod f(i) \longrightarrow \prod_{u \in I(i,j)} f(j)$$

UP product

$$S_u : \prod f(i) \xrightarrow{f(u)} f(j)$$

$$t_u : \prod f(i) \xrightarrow{\pi_j} f(j)$$

$$f(u)(-) = \pi_j(-)$$

One can check that this construction works.

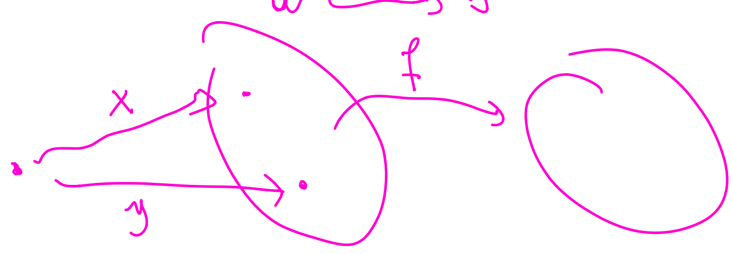
Monos

$$c \underset{t}{\overset{s}{\rightrightarrows}} a \overset{m}{\hookrightarrow} b$$

m is mono if $\forall\, s,t : \cancel{m}\, t = \cancel{m}\, s$ one has that $s = t$.

What is a mono in Set?

$$a \overset{m}{\hookrightarrow} b$$



$$f(x) = f(y) \Rightarrow x = y$$

Prop f is a mono iff $f : a \to b$

$$\begin{tikzcd} a \arrow[r, "1_a"] \arrow[d, "1_a"'] \arrow[dr, phantom, "\lrcorner", very near start] & a \arrow[d, "f"] \\ a \arrow[r, "f"] & b \end{tikzcd}$$

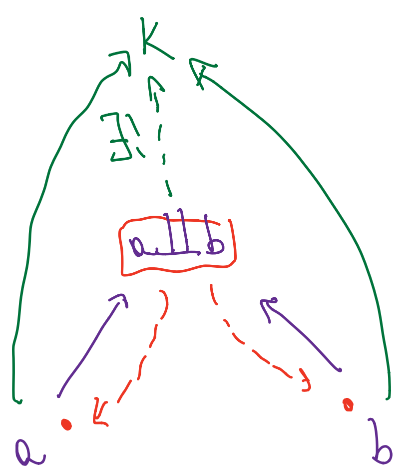
is a pullback.

Colimits

Def $I \xrightarrow{F} C$ the colimit of F ($\mathrm{colim}\, F$) is the limit of

$F^{op} : I^{op} \longrightarrow C^{op}$.

What is a coproduct so?



In Set coproducts are disjoint union.

Notational disaster

$\boxed{\lim}$ $\varprojlim$	$\boxed{\operatorname{colim}}$ $\varinjlim$
limits along inverse systems colimits along direct systems	$\boxed{\varprojlim}$ $\boxed{\varinjlim\text{colim}}$
inverse lim direct limit	limit colimit

Epimorphisms

Dual concept of mono

$$: a \xrightarrow{p} b \underset{s}{\overset{t}{\rightrightarrows}} c$$

$$t\cancel{p} = s\cancel{p}$$

Epis in Set are surj function

⚠ $\mathbb{Z} \xhookrightarrow{i} \mathbb{Q}$ in Ring is an epimorphism!

<u>Challenge</u> Characterize epi in Top.

Preservation of limits

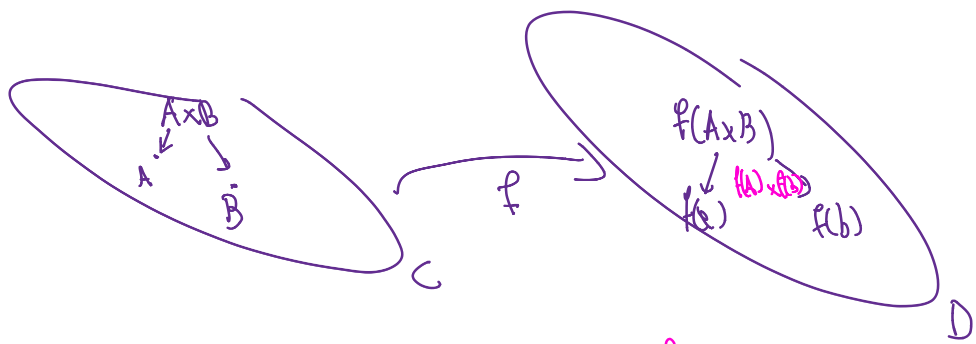
$$\begin{tikzcd} I \arrow[r, "D"] \arrow[rr, dashed, bend right, "f \circ D"'] & A \arrow[r, "f"] & B \end{tikzcd}$$

$$\begin{tikzcd} \lim D \arrow[r, "\pi_i"] \arrow[r, "\pi_j"'] \arrow[r] & D(i) \\ & D(j) \\ & D(k) \end{tikzcd} \qquad \begin{tikzcd} f(\lim D) \arrow[r, "f\pi_i"] \arrow[r, "f\pi_j"'] \arrow[r] & fD(i) \\ & fD(j) \end{tikzcd}$$

<u>Obs</u> I get a cone for $f \circ D$.

$$f(\lim D) \xrightarrow{f\pi_i}$$

<u>Def</u> f preserve $\lim D$ if $\left(f(\lim D), f(\pi_i)\right)$ is a limiting cone for $f \circ D$.



$$f(A \times B) \cong f(A) \times f(b).$$

<u>Example</u> $\mathcal{U} : \mathrm{Top} \longrightarrow \mathrm{Set}$ (forgetful functor)

$\mathcal{U}$ preserve products!
$\mathcal{U}$ " equalizer!

<u>Cor</u> it preserve all limits!

Moth: forg. functors preserve limits.

$\mathcal{U}: \boxed{\mathrm{Grp} \to \mathrm{Set}}$
$\boxed{R\text{-Mod} \to \mathrm{Set}}$