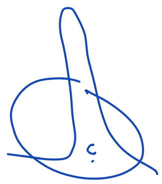
Last time

Representable functor



$$\mathcal{C}(-,c) : \mathcal{C}^{op} \longrightarrow Set.$$
$$d \longmapsto \mathcal{C}(d,c).$$

$$\mathcal{C}(c,-) : \mathcal{C} \longrightarrow Set.$$
$$d \longmapsto \mathcal{C}(c,d).$$

Yoneda embedding.

$$X \longrightarrow \mathcal{P}(X)$$
$$x \longmapsto \{x\}.$$

$$\begin{array}{ccc} X & \longrightarrow & 2^X \\ \downarrow & & \| \\ \mathcal{P}(X) & & \end{array}$$

$\mathcal{P}(X) \cong$ Discop(X).

$$\mathcal{C} \xrightarrow{\text{よ}} Set^{\mathcal{C}^{op}} \qquad \text{よ}.\ y.$$

$$\mathcal{C} \xrightarrow{\text{よ}^{\#}} (Set^{\mathcal{C}})^{op}.$$

Yoneda lemma

$$Set^{\mathcal{C}^{op}}(\text{よ}(c), F) \cong F(c).$$

<u>Cor</u> Yoneda embedding is fully faithful.

<u>Cor</u> $a \cong b \iff \text{よ}(a) \cong \text{よ}(b).$

$$\mathcal{C}(-,a) \cong \mathcal{C}(-,b).$$

Today.

Adj

- $\mathcal{C}(L-,-) \cong \mathcal{D}(-,R-)$
- Representables preserve limits.
- RAPL.
  right adjoint preserve limits
- AFT (Freyd).
  continuous functor is a right adjoint.

Limits & Adj

- Description of limits in terms of adjoints.

Limits.

- If $\mathcal{S}$ is (co)complete, $\mathcal{S}^I$ is complete too.

$$\mathrm{Quiv} \cong \mathrm{Set}^{\rightrightarrows}$$

Def

$$L : \mathcal{A} \rightleftarrows \mathcal{B} : R.$$

$$\eta : 1_{\mathcal{A}} \longrightarrow RL \qquad \text{(unit)}$$

$$\varepsilon : 1_{\mathcal{B}} \longleftarrow LR \qquad \text{(counit).}$$

+ axioms (triangle equation.)

$$\mathcal{B}(La, b) \overset{\text{nt.}}{\cong} \mathcal{A}(a, Rb).$$

Rem $L \dashv R$

$$\begin{array}{ccc} \mathcal{A}^{op} \times \mathcal{B} & \longrightarrow & \mathrm{Set}. \\ (a, b) & \longmapsto & \mathcal{B}(L(a), b). \end{array}$$

$$\begin{tikzcd} \mathcal{A}^{op} \times \mathcal{B} \arrow[r, "L^{op} \times id."] \arrow[rr, bend right, "\mathcal{B}(L-,-)."'] & \mathcal{B}^{op} \times \mathcal{B} \arrow[r, "\mathcal{B}(-,-)"] & \mathrm{Set} \end{tikzcd}$$

$$(a, b) \longmapsto (La, b) \longmapsto \mathcal{B}(La, b).$$

$$\mathcal{A}^{op} \times \mathcal{B} \longrightarrow Set$$

$$(a\ ,\ b) \longmapsto \mathcal{A}(-, R-).$$

$$\begin{tikzcd}
A^{op} \times B \arrow[r, "id^{op} \times R"] \arrow[rr, bend right, "\mathcal{A}(-, R-)."'] & A^{op} \times \mathcal{A} \arrow[r, "\mathcal{A}(-,-)"] & Set
\end{tikzcd}$$

Def $L : \mathcal{A} \rightleftarrows \mathcal{B} : R$ are adjoint if there exist a natural isomorphism.

$$\varphi : B(L-, -) \overset{\sim}{\Longrightarrow} \mathcal{A}(-, R-).$$

$$\begin{tikzcd}
\mathcal{A}^{op} \times B \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\varphi \Downarrow"] & Set
\end{tikzcd}$$

$$\begin{pmatrix} \eta \\ \varepsilon \end{pmatrix} \rightsquigarrow \varphi. \qquad \checkmark \text{ Done in lecture 3}$$

$$\varphi \overset{?}{\rightsquigarrow} \begin{pmatrix} \eta \\ \varepsilon \end{pmatrix}.$$

$$\varphi : B(L-, -) \longrightarrow \mathcal{A}(-, R-)$$

$$\frac{\eta : 1 \longrightarrow RL.}{\eta \in \mathcal{A}(x, RLx)}$$

$$\cong \varphi$$

$$B(Lx, Lx). \ni id_{L(x)}.$$

$$\eta_x := \varphi(id_{L(x)})$$

$$\varepsilon \in \mathcal{B}(LRx, x)$$
$$\cong$$
$$\mathcal{A}(Rx, Rx).$$

$$\varepsilon = \varphi^{-1}(\mathrm{id}_{Rx}).$$

"Representables preserve limits".

$\mathcal{A}$ has all small limits.

$\mathcal{A}(a,-): \mathcal{A} \longrightarrow \mathbf{Set}$ ← preserves all lims.

$$\mathcal{A}(a, \lim D) \cong \lim \mathcal{A}(a, D).$$

In the special case of products.

$$\rightsquigarrow \quad \mathcal{A}\left(a, \prod c_i\right) \cong \prod_i \mathcal{A}(a, c_i).$$

$$\varphi: \mathcal{A}(a, c \times d) \cong \mathcal{A}(a, c) \times \mathcal{A}(a, d)$$
$$f \longmapsto (\pi_d \circ f, \pi_c \circ f)$$

$$\begin{tikzcd}
 & a \arrow[d, dashed, "\exists!"] \arrow[ddl, bend right] \arrow[ddr, bend left] & \\
 & d \times c \arrow[dl] \arrow[dr] & \\
d & & c
\end{tikzcd}$$

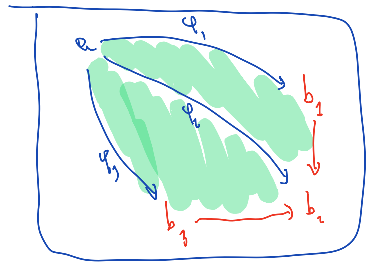
$$g \times s \longleftarrow (g, s)$$

Motto: the fact that $\mathcal{A}(a,-)$ preserves products is equivalent to the universal property of products!

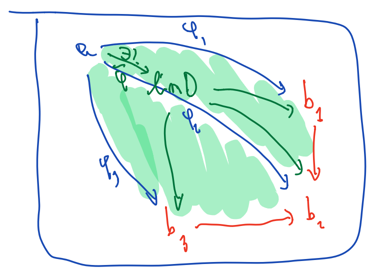
Proof of the general statement.

$$\lim_i \mathcal{A}(a, D(i)) \subset \prod_i \mathcal{A}(a, D(i))$$

$$\varphi_i : a \longrightarrow D(i).$$



$$(\varphi) \in \lim \mathcal{A}(a, D(i)).$$



$$\mathcal{A}(a, \lim D(i)) \xleftarrow{\ \sim\ } \lim \mathcal{A}(a, D(i))$$

$$\bar{\varphi} \qquad\qquad (\varphi)$$

Representables preserve limits.

$$\mathcal{A}(-,a) : \mathcal{C}^{op} \longrightarrow \mathrm{Set}$$

$$\mathcal{A}(\lim x_i, a) \cong \lim \mathcal{A}(x_i, a)$$

if I look at it in $\mathcal{C}$

$$\mathcal{A}(\mathrm{colim}\, x_i, a)$$

For contravariant representables to preserve limits is the univ. property of colimits!

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③ <u>thm</u> Right adjoints preserve limits.

$$R(\lim D) \cong \lim_i R(D(i)).$$

$$\mathcal{A}(-, R(\lim D(i))).$$

$\cong$ adjunction

$$\mathcal{B}(L-, \lim_i D(i))$$

$\cong$ representable preserve limits

$$\lim_i \mathcal{B}(L-, D(i))$$

$\cong$ adjunction

$$\lim_i \mathcal{A}(-, RD(i))$$

rep preserve limits

$$\mathcal{A}(-, \lim_i RD(i))$$

Yoneda lemma

$$R(\lim D) \cong \lim_i R(D_i)$$

Example of this story.

$$\begin{array}{ccc} \mathrm{Top} & \xrightarrow{\mathcal{U}} & \mathrm{Set} \\ (X, \tau) & \longmapsto & X \end{array}$$

$$\mathcal{U}(X \times Y) \cong \mathcal{U}(X) \times \mathcal{U}(Y)$$

$$\mathcal{U}((X \times Y, \tau_X \times \tau_Y)) \cong X \times Y.$$

<u>Rem</u> the same proof shows that left adjoints preserve COLIMITS.

(d) R preserve limits

R right adj $\Rightarrow$ R preserve limits

$\Leftarrow$?

<u>thm</u> $A \xrightarrow{R} B$ limit preserving + hypotheses $\Rightarrow$ R is a right adjoint.

(S) Adjoint functor theorem

In the case of posets

$$P \xrightarrow{f_*} Q$$

complete poset

f_* preserve infima

thm in these assumptions f_* has a left adjoint f^*.

$$f^* : Q \rightleftarrows P : f_*$$

Prof

$$f^* q \leq p \iff q \leq f_* p.$$

$f^* q :=$ inf P

$q \leq f_* p$

Limits & adjunctions

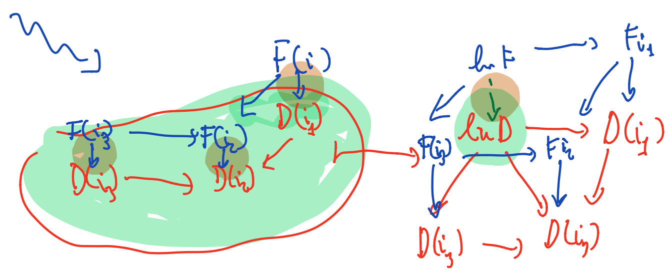
$$\mathrm{Set}^{D} = \mathrm{Set}^{\boxed{\,::\,}} \cong \mathrm{Set} \times \mathrm{Set} \xrightarrow{(-)\times(-)} \mathrm{Set}$$

$$(a\;,\;b) \longmapsto a \times b$$

Let $\mathcal{A}$ is a complete category

$$\mathcal{A}^{D} \xrightarrow{\lim} \mathcal{A}$$

$$D \longmapsto \lim D.$$



<u>thm</u> lim is a right adjoint!

$$\Delta : \mathcal{A} \rightleftarrows \mathcal{A}^{D} : \lim$$

$$a \longmapsto \Delta(a) : D \longrightarrow \mathcal{A}.$$

$$d \longmapsto a$$

$$D = \boxed{\;\cdot\;\;\cdot\;}$$

$$\Delta : \mathcal{A} \longrightarrow \mathcal{A} \times \mathcal{A}.$$

$$a \longmapsto (a, a).$$

$$\underbrace{\mathcal{A}^D(\Delta a, F)} \cong \mathcal{A}(a, \lim F).$$

$$D = \begin{tikzcd} & 1 \arrow[d] \\ 3 \arrow[r] & 2 \end{tikzcd}$$

$$\rightsquigarrow \quad \begin{tikzcd} & a \arrow[d, equal, "id_a"'] \arrow[dr] & \\ a \arrow[r, equal, "id_a"] \arrow[d] & a \arrow[dr] & F(1) \arrow[d] \\ F(3) \arrow[rr] & & F(2) \end{tikzcd} \qquad \begin{tikzcd} a \arrow[dr] \arrow[drr, bend left] \arrow[ddr, bend right] \arrow[dr, bend left] & & \\ & \lim F \arrow[r] \arrow[d] \arrow[dr] & F(1) \arrow[d] \\ & F(3) \arrow[r] & F(2) \end{tikzcd}$$

a cone
of vertex a!

$$\mathrm{colim} \dashv \Delta \dashv \lim$$

Limits in functor categories.

$G\text{-sets} \cong \mathrm{Set}^{G}$.

$\mathrm{Quiv} \cong \mathrm{Set}^{\bullet \rightrightarrows \bullet}$

- if $\mathcal{A}$ is (co)complete, then $\mathcal{A}^{I}$ is (co)complete.
- (co)limits are computed pointwise

Example

$$\mathrm{Set}^{\boxed{\bullet\ \bullet}} \cong \mathrm{Set} \times \mathrm{Set}.$$

$$(A,B) \times (C,D) \cong (A \times C,\ B \times D).$$

$$\begin{tikzcd}
 & (A \times C,\ B \times D) \arrow[dl] \arrow[dr] & \\
(A, B) & & (C, D)
\end{tikzcd}$$

Thm Let $I \xrightarrow{F} \mathcal{A}^{D}$ be a diagram.

For each $d \in D$ $\quad F_d : I \longrightarrow \mathcal{A}$
$$(-) \longmapsto F(-)(d),$$

then if all $\lim F_d$ exist then $\lim F$ exist and

$$(\lim F)(d) = \lim F_d.$$

Proof: Exercise.