SHEAVES, MANIFOLDS, COHOMOLOGY

IVAN DI LIBERTI AND ANDREA GAGNA

Exercises

salad

Exercise 1 (). Compute/Describe the category of sheaves over the one-point space and over the Sierpinski space.

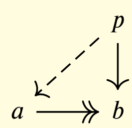
Wedhorn (). Problem 3.3.

Wedhorn (). Problem 4.4.

Wedhorn (). Problem 8.18.

categories

Exercise 2 (). Recall that an object p in a category C is *projective* if, whenever we are given a solid diagram as below, where the horizontal arrow is an epimorphism, the dashed arrow exists and makes the diagram commutative.



(a) show that p is projective if and only if $\mathsf{C}(p, -)$ preserves epimorphisms.
(b) show that in an abelian category this is equivalent to require that $\mathsf{C}(p, -)$ is exact[a].
(c) show that in the category of modules for a commutative ring, free modules are projective.
(d) show that in the category of modules for a commutative ring every object is a quotient of a projective one.

Exercise 3 (). Let $R : \mathsf{A} \to \mathsf{B}$ be a right adjoint between complete categories. Show that:

(a) R preserves monomorphisms.
(b) if R is conservative, then it is faithful.
(c) if B is **Set**, then there exists an $a \in \mathsf{A}$ such that $R \cong \mathsf{A}(a, -)$.
(d) a is a generator if and only if $\mathsf{A}(a, -)$ is faithful.

[a] preserves both finite limits and colimits.

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sheaves

Wedhorn (🔋). Problem 3.17.

Exercise 4 (🔋, ☕). Fundamental theorem of covering spaces. Inspired by the equivalence $\mathsf{Et}(X) \simeq \mathsf{Sh}(X)$ between sheaves and étalé spaces, define the category of coverings $\mathsf{Cov}(X)$ over X and prove that it is equivalent to the category of sets equipped with a $\Pi_1(X)$-action, where $\Pi_1(X)$ is the fundamental groupoid of X,

$$\mathsf{Cov}(X) \simeq \mathbf{Set}^{\Pi_1(X)}.$$

ringed spaces

Wedhorn (🔋). Problem 4.5.

Wedhorn (🔋). Problem 4.6.

bundles

Wedhorn (🔋). Problem 8.6.

Wedhorn (🔋). Problem 8.7.

Wedhorn (🔋). Problem 8.9.

skyscrapers

Wedhorn (🔋, ☕). Problem 3.6.

Wedhorn (🔋). Problem 7.5.

Wedhorn (🔋). Problem 8.3.

The riddle (⚠, ☕). Give a tentative definition of Hausdorff topos. Show that the topos of sheaves over a Hausdorff locally compact space is Hausdorff.
Hint. you will need to define closed geometric morphisms and product of topoi.

- 🔴 the exercises in the red group are mandatory.
- 🟡 pick at least one exercise from each of the yellow groups.
- 🟤 nothing is mandatory in the brown box.
- 🟢 The riddle. It's just there to let you think about it. It is not a mandatory exercise, nor it counts for your evaluation. Yet, it has a lot to teach.
- ☕ useful to deepen your understanding. Take your time to solve it. (May not be challenging at all.)
- 🔋 measures the difficulty of the exercise. A technically easy exercise is still very important for the foundations of your knowledge.
- ⚠ It's just too hard.

The label **Wedhorn** refers to the book **Manifolds, Sheaves, and Cohomology**, by *Wedhorn*.