

Last time

Limits

- Product
- Equalizers
- Pull back.
- General def.

Colimits

- Coprod
- Coequalizer
- Pushout
- General def.

Monos

Epimorphisms

Exa

- Product of sets
- \wedge in posets
- Kernels
- Quotients
- Monos in sets are injective maps
- Epi in sets are surjective maps
- $\mathbb{Z} \hookrightarrow \mathbb{Q}$ is an epi in Ring.

thm all limits $\Leftrightarrow \prod$ + equalizers

all colimits $\Leftrightarrow \coprod$ + coequalizers

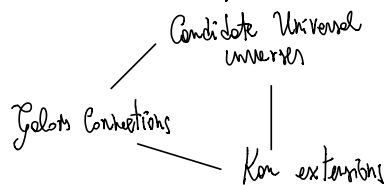
Lecture 3: Adjunctions

Two approaches to adjunctions

Empirical / Historical

"free constructions in universal algebras"

Conceptual / In retrospect



First story (free groups) $\text{Grp}(F(X), G) \cong \text{Set}(X, \text{UG})$
 Second story (τ^{-1} as a generalized inverse).

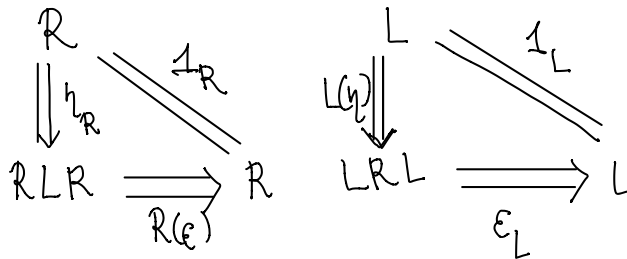
Def Let $L: A \rightleftarrows B: R$ be functors.

We say that L is left adjoint to R ($L \dashv R$) if there exist natural transformations

(unit) $\eta: 1_A \implies RL$

(counit) $\epsilon: LR \implies 1_B$

triangle equations

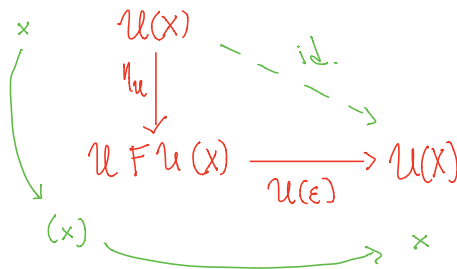


"Mathematically precise definition"

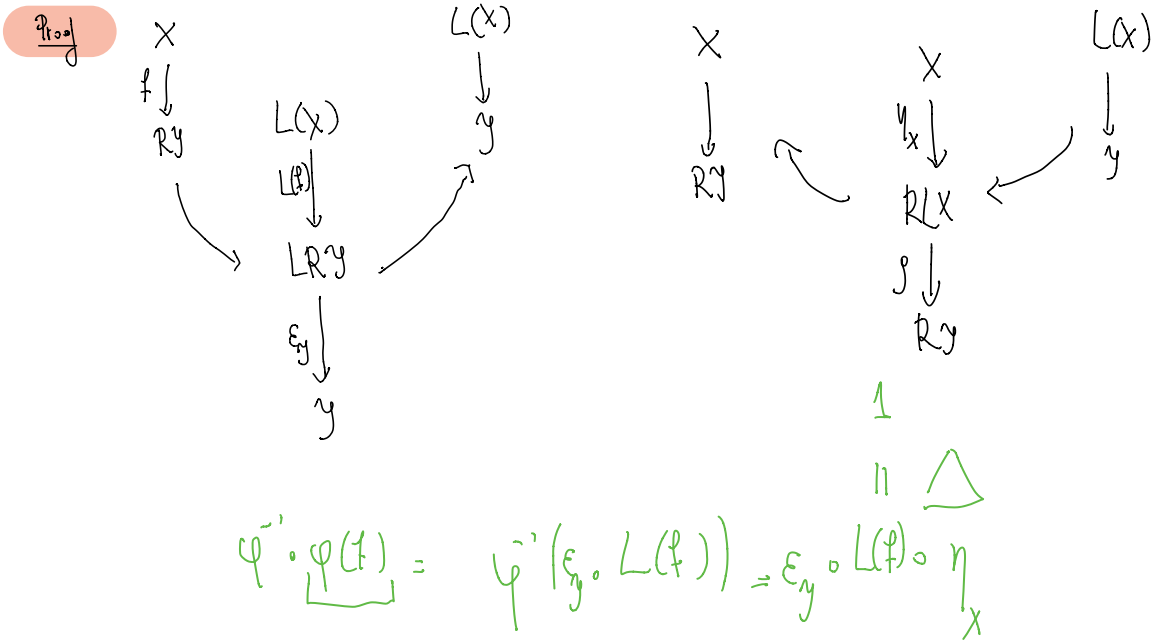
this is very hard to prep. sanity check.

$$F : \text{Set} \rightleftarrows \text{Grp} : U$$

$$\begin{array}{ccc|ccc} FU(G) & \xrightarrow{\epsilon_G} & G & & X & \xrightarrow{\eta_X} & UF(X) \\ (w) & \longmapsto & w & & x & \longmapsto & g_x \end{array}$$



Thm $L : A \rightleftarrows B : R$, then
 $\varphi : A(\underline{x}, R\underline{y}) \rightleftarrows B(L\underline{x}, \underline{y}) : \varphi^{-1}$



The bijection above is "natural" in A and B^P .

Other examples

- Vect_k Set
- Grp Set
- Ab Set
- Ab Grp
- $\text{Top} \hookrightarrow \text{Set}$
- cartesian closed category -
- closure operators.
- pointed sets -
- monoidal structure on Ab -

Non example

Fld Set

Remark Adjunctions compose

CATEGORY THEORY

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EXERCISES

Leinster (□). 2.1.12

Leinster (□). 2.1.15

Leinster (■). 2.1.16

Leinster (□). 2.2.10

Leinster (□). 2.2.11

Leinster (■). 2.2.12

Exercise 1 (■). Show that the a right adjoint preserves monomorphisms.

Leinster (■). 2.3.12

Exercise 2 (■). What is a cartesian closed bounded lattice?

Exercise 3 (■). Show that the inclusion of the category of abelian groups in the category of groups $\iota : \text{Ab} \hookrightarrow \text{Grp}$ has a left adjoint.

Exercise 4 (■). Prove that the inclusion $\text{Haus} \hookrightarrow \text{Top}$ of the full subcategory of Hausdorff spaces into the category of all spaces has a left adjoint.

- the exercises in the red group are mandatory.
 - pick at least one exercise from each of the yellow groups.
 - pick at least two exercises from each of the blue groups.
 - nothing is mandatory in the brown box.
 - The riddle of the week. It's just there to let you think about it. It is not a mandatory exercise, nor it counts for your evaluation. Yet, it has a lot to teach.
 - ☞ useful to deepen your understanding. Take your time to solve it. (May not be challenging at all.)
 - measures the difficulty of the exercise. Note that a technically easy exercise is still very important for the foundations of your knowledge.
 - ▲ It's just too hard.
- The label **Leinster** refers to the book **Basic Category Theory**, by *Leinster*.
The label **Riehl** refers to the book **Category Theory in context**, by *Riehl*.