Last time

Limits
- Product
- Equalizers
- Pull back.
- General def.

Colimits
- Coprod
- Coequalizer
- Push out
- General def

Monos

Epimorphisms

Exa
- Product of sets
- $\wedge$ in posets
- Kernels
- Quotients
- Mons in sets are injective maps
- Epi in sets are surjective maps
- $\mathbb{Z} \hookrightarrow \mathbb{Q}$ is an epi in Ring.

thm all limits $\Leftrightarrow$ $\prod$ + equalizers

all colimits $\Leftrightarrow$ $\coprod$ + coequalizers

Lecture 3: Adjunctions

Two approaches to adjunctions

Empirical / Historical

"Free constructions in universal algebras"

Conceptual / In retrospect

Candidate Universal inverses

Galois Connections

Kan extensions

<u>First story</u> (free groups) $\quad Grp(F(X), G) \cong Set(X, U(G))$

<u>Second story</u> (f^{-1} as a generalized inverse).

"Mathematically precise definition"

Def Let $L : A \rightleftarrows B : R$ be functors. We say that **L is left adjoint to R** ($L \dashv R$) if there exist natural transformations

$$\text{(unit)} \qquad \eta : 1_A \Longrightarrow RL$$

$$\text{(counit)} \qquad \varepsilon : LR \Longrightarrow 1_B$$

triangle equalities

$$\begin{tikzcd} R \arrow[d, Rightarrow, "\eta_R"] \arrow[dr, equal, "1_R"] & \\ RLR \arrow[r, Rightarrow, "R(\varepsilon)"'] & R \end{tikzcd} \qquad \begin{tikzcd} L \arrow[d, Rightarrow, "L(\eta)"'] \arrow[dr, equal, "1_L"] & \\ LRL \arrow[r, Rightarrow, "\varepsilon_L"'] & L \end{tikzcd}$$

this is very hard to grasp. Sanity check.

$$F : \mathrm{Set} \rightleftarrows \mathrm{Grp} : \mathcal{U}$$

$$\begin{tikzcd} F\mathcal{U}(G) \arrow[r, "\varepsilon_G"] & G \\ (w) \arrow[r, mapsto] & w \end{tikzcd} \qquad \Bigg| \qquad \begin{tikzcd} X \arrow[r, "\eta_X"] & \mathcal{U}F(X) \\ x \arrow[r, mapsto] & g_x \end{tikzcd}$$

$$\begin{tikzcd} x \arrow[d, bend right, mapsto] & \mathcal{U}(X) \arrow[d, "\eta_{\mathcal{U}}"'] \arrow[dr, dashed, "id."] & \\ & \mathcal{U}F\mathcal{U}(X) \arrow[r, "\mathcal{U}(\varepsilon)"'] & \mathcal{U}(X) \\ (x) \arrow[rr, bend right, mapsto] & & x \end{tikzcd}$$

Thm $L : A \rightleftarrows B : R$, then

$$\varphi : A(\underline{X}, R\underline{Y}) \overset{\sim}{\rightleftarrows} B(L\underline{X}, \underline{Y}) : \varphi^{-1}$$

Proof

$$\begin{tikzcd} X \arrow[d, "f"'] \\ RY \end{tikzcd} \longmapsto \begin{tikzcd} L(X) \arrow[d, "L(f)"'] \\ LRY \arrow[d, "\varepsilon_y"'] \\ y \end{tikzcd} \longmapsto \begin{tikzcd} L(X) \arrow[d] \\ y \end{tikzcd}$$

$$\begin{tikzcd} X \arrow[d] \\ RY \end{tikzcd} \longleftarrow \begin{tikzcd} X \arrow[d, "\eta_X"'] \\ RLX \arrow[d, "\rho"'] \\ Ry \end{tikzcd} \longleftarrow \begin{tikzcd} L(X) \arrow[d] \\ y \end{tikzcd}$$

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$\shortparallel \quad \triangle$

$$\varphi^{-1} \circ \underbrace{\varphi(f)} = \varphi^{-1}(\varepsilon_y \circ L(f)) \Rightarrow \varepsilon_y \circ L(f) \circ \eta_X$$

The bijection above is "natural in A and B".

<u>Other examples</u>

- Vect_k $\quad$ Set
- Grp $\quad$ Set
- Ab $\quad$ Set
- Ab $\quad$ Grp
- Top $\quad$ Set
- cartesian closed category.
- Closure operators.
- Pointed sets.
- Monoidal structure on Ab.

Non example

Fld $\quad$ Set

<u>Rem</u> Adjunctions compose

CATEGORY THEORY

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Exercises

Leinster ( ). 2.1.12

Leinster ( ). 2.1.15

Leinster ( ). 2.1.16

Leinster ( ). 2.2.10

Leinster ( ). 2.2.11

Leinster ( ). 2.2.12

Exercise 1 ( ). Show that the a right adjoint preserves monomorphisms.

Leinster ( ). 2.3.12

Exercise 2 ( ). What is a cartesian closed bounded lattice?

Exercise 3 ( ). Show that the inclusion of the category of abelian groups in the category of groups $\imath : \mathsf{Ab} \hookrightarrow \mathsf{Grp}$ has a left adjoint.

Exercise 4 ( ). Prove that the inclusion $\mathrm{Haus} \hookrightarrow \mathrm{Top}$ of the full subcategory of Hausdorff spaces into the category of all spaces has a left adjoint.

- the exercises in the red group are mandatory.
- pick at least one exercise from each of the yellow groups.
- pick at least two exercises from each of the blue groups.
- nothing is mandatory in the brown box.
- The riddle of the week. It's just there to let you think about it. It is not a mandatory exercise, nor it counts for your evaluation. Yet, it has a lot to teach.
- useful to deepen your understanding. Take your time to solve it. (May not be challenging at all.)
- measures the difficulty of the exercise. Note that a technically easy exercise is still very important for the foundations of your knowledge.
- It's just too hard.

The label **Leinster** refers to the book **Basic Category Theory**, by *Leinster*.
The label **Riehl** refers to the book **Category Theory in context**, by *Riehl*.

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