Last time

Representables $\mathcal{C}(-,c) : \mathcal{C}^{op} \longrightarrow Set$
$d \longmapsto \mathcal{C}(d,c)$.

Yoneda embedding (bump functions). $\quad \text{よ} : \mathcal{C} \longrightarrow Set^{\mathcal{C}^{op}}$
$c \longmapsto \mathcal{C}(-,c)$.

Yoneda lemma $\quad Set^{\mathcal{C}^{op}}(\text{よ}(c), F) \cong F(c)$.

Cor the Yoneda embedding is fully faithful.

Cor $\quad a \cong b \iff \text{よ}a \cong \text{よ}b$.
(objects are determined by their behavior).

Today: Interactions between Yoneda, limits, adjunctions.

Adjunctions

- Presentation of adjoint functors informed of the Yoneda embedding
- Representables preserve limits.
- Left adjoint preserve colimits. Right adjoint preserve limits
- the adjoint functor theorem

Limits & Adjunctions.

- Description of limits in terms of adjoints.

Limits

- If $\mathcal{S}$ is (co)complete, so is $\mathcal{S}^A$.
- A description of (co)limits in $\mathcal{S}^A$.

Adjunctions

a) In previous lecture we were not completely happy with our def of adj.

Def $L : \mathcal{A} \rightleftarrows \mathcal{B} : R$

$\eta : 1_{\mathcal{A}} \longrightarrow RL$ (unit)

$\varepsilon : 1_{\mathcal{B}} \longleftarrow LR$. (counit)

+ axioms (triangle equalities).

Good definition, but we had another intuition.

$$\mathcal{B}(L(a), b) \cong \mathcal{A}(a, Rb).$$

We can now turn this into a definition.

For an adj $L \dashv R$ as above, we define

$$\begin{aligned} \mathcal{A}^{op} \times \mathcal{B} &\longrightarrow Set \\ a, b &\longmapsto \mathcal{B}(L(a), b). \end{aligned}$$

this is a functor because it is

$$\begin{tikzcd} \mathcal{A}^{op} \times \mathcal{B} \arrow[r, "L^{o} \times id"] \arrow[rr, bend right, "\mathcal{B}(L-,-)"'] & \mathcal{B}^{o} \times \mathcal{B} \arrow[r, "\mathcal{B}(-,-)"] & \mathrm{Set} \end{tikzcd}$$

similarly define

$$\begin{aligned} \mathcal{A}^{op} \times \mathcal{B} &\longrightarrow \mathrm{Set} \\ (a, b) &\longmapsto \mathcal{A}(a, Rb) \end{aligned}$$

this is again a functor.

$$\begin{tikzcd} \mathcal{A}^{op} \times \mathcal{B} \arrow[r, "id^{op} \times R"] \arrow[rr, bend right, "\mathcal{A}(-,R-)"'] & \mathcal{A}^{o} \times \mathcal{A} \arrow[r, "\mathcal{A}(-,-)"] & \mathrm{Set} \end{tikzcd}$$

<u>Def</u> $L : \mathcal{A} \rightleftarrows \mathcal{B} : R$ are adjoint iff there exists a natural isomorphism

$$\varphi : \mathcal{B}(L-,-) \overset{\cong}{\Rightarrow} \mathcal{A}(-,R-).$$

We have essentially already seen how to find φ from (η, ε).

No we see how to do the opposite

$$\varphi \rightsquigarrow \begin{pmatrix} \eta \\ \varepsilon \end{pmatrix}.$$

Nothing more easy.

$\eta : 1 \longrightarrow RL.$

$\eta \in \mathcal{A}(-, RL).$

$\mathcal{B}(L-, L-) \cong \mathcal{A}(-, RL-)$

$\eta_{(-)} := \varphi(\mathrm{id}_{L(-)}).$

(b) Le $\mathcal{A}$ be a category with all limits and colimits. We will show that

$$\mathcal{A}(a, -) : \mathcal{A} \longrightarrow \mathrm{Set}$$

preserve all limits.

For example, in the case of products,

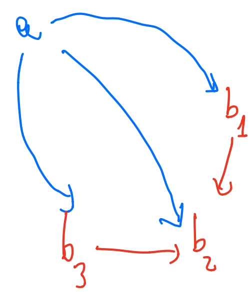
$$\mathcal{A}(a, \textstyle\prod b_i) \cong \prod \mathcal{A}(a, b_i).$$

but this is precisely the universal property of the product!

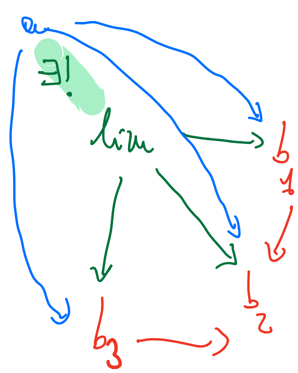
The same argument works in general.
The property of preserving limits is precisely the same of being a limit.

$$\lim \mathcal{A}(a, b_i) \hookrightarrow \prod \mathcal{A}(a, b_i)$$
$$\ni x$$

x is a coherent family of arrows $x_i \in \mathcal{A}(a, b_i)$ that "commutes with the diagram".



this correspond $1 \leftrightarrow 1$ to maps into the limit!



$$\mathcal{A}(a, \lim D) \simeq \lim \mathcal{A}(a, D_i)$$

Of course the same is true for

$$\mathcal{A}(-,b) : \mathcal{A}^{op} \longrightarrow Set.$$

ⓒ Right adjoints preserve limits.

$$\begin{aligned}
\mathcal{A}(-, R(\lim D)) &\cong \mathcal{B}(L-, \lim D) \\
&\cong \lim \mathcal{B}(L-, D) \\
&\cong \lim \mathcal{B}(-, RD) \\
&\cong \mathcal{A}(-, \lim RD).
\end{aligned}$$

By Yoneda

$R \lim D \cong \lim RD$

d

Is the converse true?

$\boxed{\text{the adjoint functor theorem.}}$

thm Let $A \xrightarrow{R} B$ be a functor preserving limits + assumptions. Then it is a right adjoint.

forgetful

the example of fresh!

the case of posets ... (the explicit construction).

Limits and adjunctions

Say the A has all limits of shape I.
For example, Set has all products.

$$\begin{tikzcd} \mathrm{Set}^{\bullet\bullet} \arrow[drr, "(-)\times(-)"] \arrow[d, phantom, "\cong" rotate=90] & & \\ \mathrm{Set}\times\mathrm{Set} \arrow[rr] & & \mathrm{Set} \\ (a, b) \arrow[rr, mapsto] & & a\times b \end{tikzcd}$$

$$\mathcal{A}^{D} \overset{?}{\underset{\lim}{\rightsquigarrow}} \mathcal{A}$$

$$D \longmapsto \lim D .$$

$\boxed{\text{this functor is a right adjoint.}}$

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{\Delta} & \mathcal{A}^{D} \\ a & \longmapsto & \Delta(a) : D \longrightarrow \mathcal{A} \\ & & \quad d \longmapsto a \\ & & \quad f \longmapsto id_a \end{array}$$

$$\Delta : \mathcal{A} \rightleftarrows \mathcal{A}^{D} : \lim$$

$$\mathcal{A}^{D}(\Delta a, F) \cong \mathcal{A}(a, \lim F)$$

! ||

Cone F vertex a !

this is a ~~key~~ example of adjunction for us

Limits in functor categories.

Functor categories are nice. Two examples.

G - sets | $Set^{\cdot \rightrightarrows \cdot} \cong Quiv.$

We show by abstract nonsense that they are likely to be complete.

$\boxed{\text{Thm If } \mathcal{A} \text{ is complete and } I \text{ is small, } \mathcal{A}^I \text{ is complete}}$

Cor Colimits of Quivers exist.

$\boxed{\text{Thm And are computed pointwise!}}$

Example. We work in $Set^{\cdot\ \cdot} \cong Set \times Set$.
We want to compute

$(A, B) \times (C, D)$. this is $(A \times C, B \times D)$.

The precise theorem is the following.

Let $I \xrightarrow{F} \mathcal{A}^D$ be a diagram. And for each $d \in D$ consider

$$\begin{aligned} F_d : I &\longrightarrow \mathcal{A} \\ (-) &\longmapsto F(-)(d). \end{aligned}$$

Then if all $\lim F_d$ exist, then

$\lim F$ exist and $\lim F(d) = \lim F_d$.

CATEGORY THEORY

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Exercises

Leinster ( ). 6.2.20

Leinster ( ). 6.2.21

Leinster ( ). 6.3.21(a)

Leinster ( ). 6.3.22

Leinster ( ). 6.3.26

Leinster ( ). 6.3.27

- the exercises in the red group are mandatory.
- pick at least one exercise from each of the yellow groups.
- pick at least two exercises from each of the blue groups.
- nothing is mandatory in the brown box.
- The riddle of the week. It's just there to let you think about it. It is not a mandatory exercise, nor it counts for your evaluation. Yet, it has a lot to teach.
- useful to deepen your understanding. Take your time to solve it. (May not be challenging at all.)
- measures the difficulty of the exercise. Note that a technically easy exercise is still very important for the foundations of your knowledge.
- It's just too hard.

The label **Leinster** refers to the book **Basic Category Theory**, by *Leinster*.
The label **Riehl** refers to the book **Category Theory in context**, by *Riehl*.

Date: November 10, 2020.