Categories, functors, natural transformation

<u>Def</u> (Category). A cat $\mathcal{A}$ consists of

- a collection of objects $ob(\mathcal{A})$.
- for all $a, b \in \mathcal{A}$ we have a set $\underbrace{\mathcal{A}(a,b)}$, $\underset{\mathcal{A}}{\underline{Hom}}\underline{(a,b)}$
- $\forall$ a, b, c a function

$$\circ : \mathcal{A}(a,b) \times \mathcal{A}(b,c) \longrightarrow \mathcal{A}(a,c)$$

composition

$$\left\{\begin{array}{l} \cdot\ \circ \text{ is } \underline{\text{associative}} \\ \quad \underline{(h \circ g) \circ f} = h \circ (g \circ f). \\ \\ \cdot\ \forall\ a \in \mathcal{A} \text{ there is an element} \\ \quad 1_a \in \mathcal{A}(a,a) \end{array}\right.$$

$$\left.\begin{array}{l} (1_a \circ h) = h \\ (h \circ 1_a) = h. \end{array}\right\} \text{identity law}$$

<u>Ex</u>

<u>Set</u> $\to$ Ob(Set) are sets

<u>Set</u> $\to$ a, b sets $\quad$ Set(a,b) = {functions $a \to b$}.

locally small categories

$\mathcal{A}(a,b)$ set

locally finite category.

$\mathcal{A}(a,b)$ is finite $\forall$ a, b.

$\underline{\text{FinSet}}$ → Obj finite sets

→ a, b FinSet(a, b) = {functions a → b}

$\underline{\text{Top}}$ → ob = topological spaces

→ Top(a, b) = continuous functions.

$\circ$: $\text{Top}(a,b) \times \text{Top}(b,c) \longrightarrow \text{Top}(a,c)$

$f \quad , \quad g \longmapsto \underline{g \circ f}$

1: $X \xrightarrow{id} X$

$\underline{\text{Grp}}$ → groups

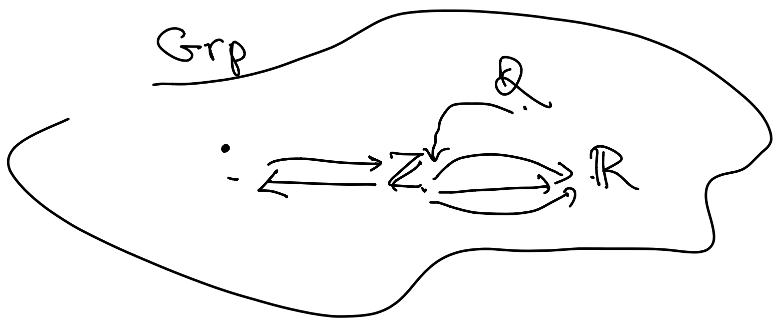
→ group homomorphism

R ring $\quad$ R-Mod → R-mod,

$\mathbb{K}$-Vect → linear maps

Rings
- → rings
- → ring homos

All the examples above are math structures



Π theory

$\underline{\Pi\text{-Mod}}$

elementary embedding

homomorphism

$\mathcal{C}$ $\begin{cases} - \text{ structured sets} \\ - \text{ decorated function} \end{cases}$

Set → $\mathcal{C}$, Ring → $\mathcal{C}$, Grps → $\mathcal{C}$, Top → $\mathcal{C}$

Ex Let $\mathbb{P}$ be a poset $(\mathbb{P}, \leq)$

$$\mathbb{P} \begin{cases} \longrightarrow Ob(\mathbb{P}) = \text{elements of } \mathbb{P} \\ \longrightarrow \mathbb{P}(a,b) = \begin{cases} \top & \text{if } a < b \\ \bot & \text{otherwise} \end{cases} \end{cases}$$

$$\mathbb{P}(a,b) \times \mathbb{P}(b,c) \longrightarrow \mathbb{P}(a,c)$$

$\top$	$\top$	$\longmapsto$	$\top$
$\top$	$\bot$	$\longmapsto$	$\bot$
$\bot$	$\bot$	$\longmapsto$	$\bot$
$\bot$	$\top$	$\longmapsto$	$\bot$

Let X be a topological space

$$\mathcal{O}(X) = \begin{cases} \text{open sets} \\ \\ \underline{a \to b \iff a \subset b} \end{cases}$$

Let $\mathcal{C}$ be a category w/ 1 object.

$\bullet_c$ $\quad \underline{\mathcal{C}(c,c)}$ monoid $\longrightarrow \begin{cases} \circ : \mathcal{C}(c,c) \times \mathcal{C}(c,c) \longrightarrow \mathcal{C}(c,c) \\ 1 \in \mathcal{C}(c,c). \end{cases}$

Every monoid M can be seen as a 1-object category

$$\mathbb{M} \begin{cases} \to \bullet c \\ \to M(c,c) = M. \end{cases}$$

<u>Ex</u> $\quad \mathrm{Cat} \begin{cases} \to \text{small categories} \\ \to \text{functors} \end{cases}$

<u>Ex</u> $\quad \mathrm{Pos} \begin{cases} \to \text{posets} \\ \to \text{monotone functions} \end{cases}$

<u>Dual category</u>

$\mathcal{C}$ category $\rightsquigarrow \mathcal{C}^{op}$ (opposite of $\mathcal{C}$) (dual of $\mathcal{C}$)

$$\mathcal{C}^{op} \begin{cases} \to \mathrm{ob} = \mathrm{ob}(\mathcal{C}). \\ \to \mathcal{C}^{op}(a,b) = \mathcal{C}(b,a) \end{cases}$$

$a \to b := b \to a.$

$\underline{\mathbb{P}}$ category associated to a poset

$\underline{\mathbb{P}^{op}}$ is the category associated to the poset P^{op}

$a \leq_{P^{op}} b \iff b \leq_P a.$

Set^{op}

<u>Functors</u> (is a way to compare categories).

Functors | Categories = functions | Set.

<u>Def</u> (Functor)
Given two categories $\mathcal{A}, \mathcal{B}$, a functor is

- a class function $F: ob(\mathcal{A}) \longrightarrow ob(\mathcal{B})$
- $A, A' \in \mathcal{A}$ a function

$$\mathcal{A}(A, A') \longrightarrow \mathcal{B}(F(A), F(A'))$$

$$\begin{array}{ccc} a & & F(a) \\ f\downarrow & \longmapsto & \downarrow F(f) \\ b & & F(b) \end{array}$$

- $F(f' \circ f) = F(f') \circ F(f)$ & $F(1_a) = 1_{F(a)}$

$$\begin{tikzcd} a \arrow[d, "f"] \arrow[dd, bend right=60, "g \circ f"'] \\ b \arrow[d, "g"] \\ c \end{tikzcd} \qquad \begin{tikzcd} F(a) \arrow[d, "F(f)"'] \arrow[dd, bend left=60, "F(g \circ f)"] \\ F(b) \arrow[d, "F(g)"'] \\ F(c) \end{tikzcd}$$

Examples

$1 : \mathrm{Set} \longrightarrow \mathrm{Set}$

$a \longmapsto a$

$f \longmapsto f$.

$- \xrightarrow{\ \mathcal{U}\ } \mathrm{Set}$

Forgetful

$(X, \tau) \longmapsto X$

$f \longmapsto f$

Set

$\mathrm{Grp} \longrightarrow$

$\mathrm{Mod} \longrightarrow \mathrm{Set}$

$\mathrm{Ring} \longrightarrow \mathrm{Set}$

$\mathrm{Ab} \longrightarrow \mathrm{Grp}$

$A \longmapsto A$

$R\text{-}\mathrm{Mod} \longrightarrow \mathrm{Set}$

Abelianization functor

$\mathrm{Grp} \longrightarrow \mathrm{Ab}$

$$\begin{tikzcd} G \arrow[r, mapsto] \arrow[d, "f"'] & G/[G,G] \arrow[d, "\bar f"] \\ H & H/[H,H] \end{tikzcd}$$

$$
\begin{tikzcd}
G \arrow[r, "f"] \arrow[d] & H \arrow[d] \\
G/[G,G] \arrow[r, dashed, "\bar{f}"] & H/[H,H]
\end{tikzcd}
$$

$$\pi_1 : \quad Top_* \longrightarrow Grp$$

is the category of pointed spaces (X, x_0)

$f : (X, x_0) \longrightarrow (Y, y_0)$ is a continuous function s.t. $f(x_0) = y_0$.

$$
\begin{tikzcd}
X, x_0 \arrow[r, mapsto] \arrow[d, "f"'] & \pi_1(X, x_0) \arrow[d, "{[f]}"'] \\
{} & {}
\end{tikzcd}
$$

<u>Free functors</u>

$$
\begin{tikzcd}
Set \arrow[r, "F"] & Vect_{\mathbb{K}} & \\
X \arrow[r, mapsto] \arrow[d, "f"'] & \mathbb{K}[X] = \left\{ \sum_{k_i \in \mathbb{K}} k_i x_i \right\} \arrow[d, "F(f)"'] & \\
Y & \mathbb{K}[Y] & \sum k_i f(x_i)
\end{tikzcd}
$$

$$\mathbf{Set} \longrightarrow \mathbf{Ab}$$
$$X \longmapsto \mathbb{Z}[X] = \left\{ \sum n_i x_i \right\}$$

$$\mathbf{Set} \longrightarrow \mathbf{Grp}$$
$$X \longmapsto F_X = \{ x_1 x_2 x_3 \}$$

$$\begin{tikzcd}
\mathbf{Top} \arrow[r, "u"] & \mathbf{Set} \arrow[l, bend right, "\text{disc}"'] \arrow[l, bend left, "\text{ind}"]
\end{tikzcd}$$

$$\mathbf{Set} \longrightarrow \mathbf{Top}$$
$$X \longmapsto (X, \text{disc})$$
$$f \longmapsto f$$

$$X \longmapsto (X, \text{ind})$$
$$f \longmapsto f.$$

$$\mathbf{Pos} \xrightarrow{\ i\ } \mathbf{Cat}$$ ← categories, functors

$$P \longmapsto \mathbb{P}$$ ← obj are elements of P

$f \downarrow \qquad f \downarrow$ ← $\mathbb{P}(a,b)$ = 1 if $a \leq b$; 0 if not

$$Q \qquad \mathbb{Q}$$ is functorial

$f, g \mid g \circ f$

$$\mathbf{Cat} \xrightarrow{\ (\ )/\sim\ } \mathbf{Pos}$$
$$\mathcal{C} \longmapsto (\mathcal{C}/\sim)$$
→ elements are objects of $\mathcal{C}$
→ $a \leq b \iff \mathcal{C}(a,b) \neq \emptyset$.

$$\begin{tikzcd}
\mathbb{C} \arrow[r, bend left] & \mathbb{Q} \arrow[l] \arrow[d] & \\
& \mathbb{Z} \arrow[ul] & \mathbb{R} \arrow[ul] \arrow[l]
\end{tikzcd}$$

Grp

$a \leq b \leq a$

$a = b$

$$\mathbb{Z} \simeq \mathbb{R} \simeq \mathbb{Q} \simeq \mathbb{C}$$

$$\begin{tikzcd}
\mathrm{Pos} \arrow[r, "i"] \arrow[rr, bend right, "\mathrm{id}_{\mathrm{Pos}}"'] & \overset{iP}{\mathrm{Cat}} \arrow[r, "(\ )/\sim"] \arrow[d, phantom, "\Downarrow"] & \mathrm{Pos} \\
& {} &
\end{tikzcd}$$

√ contravariant functors

- faithful & conservative functors ⊳ full
- natural transformations

X a space, a manifold

$\mathcal{O}(X)$ = the category of open sets

$$\begin{tikzcd}
U \arrow[r, mapsto] & \mathrm{Cont}(U, \mathbb{R}) & f \circ i \\
V \arrow[u, phantom, "\cap"] & \mathrm{Cont}(V, \mathbb{R}) \arrow[u, "i^*"'] & f \arrow[u, mapsto]
\end{tikzcd}
\qquad
\begin{tikzcd}
U \arrow[r, "i"] \arrow[rr, bend right, "f \circ i"'] & V \arrow[r, "f"] & \mathbb{R}
\end{tikzcd}$$

$$\begin{tikzcd}
F(U) & U \arrow[d, "i"] \\
F(V) \arrow[u, "i^*"] & V
\end{tikzcd}$$

$$\mathcal{O}(X)^{op} \longrightarrow \mathrm{Set}$$

opposite categories were invented to deal with contravariant assignments

$f : \mathcal{A} \to \mathcal{B}$

Def A functor is faithful when $\forall\, a, b$

$$f_{a,b} : \underbrace{\mathcal{A}(a, b)}_{} \longrightarrow \underbrace{\mathcal{B}(f(a), f(b))}_{}$$

is injective.

Ex $\mathcal{U} : \mathsf{Grp} \longrightarrow \mathsf{Sets}$

$$\mathrm{Hom}_{\mathsf{Grp}}(G, H) \underset{\neq}{\subseteq} \mathrm{Func}(G, H)$$

Vect

Forgetful functors are faithful

Def Full function

$$f : \mathcal{A}(a, b) \longrightarrow \mathcal{B}(f(a), f(b))$$

is surjective.

$\mathcal{U} : \mathsf{Grp} \longrightarrow \mathsf{Set}$

Forg functors are not full.

Def (Natural transformation) Given two functors

$$\begin{tikzcd} \mathcal{A} \arrow[r, bend left, "f"{name=F}] \arrow[r, bend right, "g"'{name=G}] \arrow[from=F, to=G, phantom, "\Downarrow\alpha"] & \mathcal{B} \end{tikzcd}$$

is a family of $\mathcal{B}$ - morphism such that.

$\underline{\alpha} : f(-) \to g(-)$

$$\begin{tikzcd} \mathcal{A} \\ a \arrow[d, "h"'] \\ b \end{tikzcd} \qquad \begin{tikzcd} f(a) \arrow[r, "\alpha_a"] \arrow[d, "f(h)"'] \arrow[dr, phantom, "="] & g(a) \arrow[d, "g(h)"] \\ f(b) \arrow[r, "\alpha_b"'] & g(b) \end{tikzcd}$$

Such that $\forall h$ the diagram above commutes

Exam

$$\begin{tikzcd} \mathrm{Grp} \arrow[rr, "id"{name=I}] \arrow[rr, bend right, "(-)/[-,-]"'{name=Q}] \arrow[from=I, to=Q, phantom, "\pi\Downarrow"] & & \mathrm{Grp} \end{tikzcd}$$

$$\begin{tikzcd} H \arrow[d, "f"'] \\ G \end{tikzcd} \qquad \begin{tikzcd} H \arrow[r, "f"] \arrow[d, "\pi_H"'] \arrow[dr, phantom, "="] & G \arrow[d, "\pi_G"] \\ H/[H,H] \arrow[r, "\bar{f}"'] & G/[G,G] \end{tikzcd}$$

Last example $\boxed{G\text{-sets}}$

A Group G is a category w/ 1-object

$$\mathbf{Set}^G \begin{cases} \text{Functors } G \longrightarrow \mathbf{Set} \\ \mathbf{Set}^G(f, g) = \text{Natural transformations from } f \text{ to } g. \end{cases}$$

$$\begin{tikzcd} G \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow"] & \mathbf{Set} \end{tikzcd}$$

$$G\text{-}\mathbf{Set} \begin{cases} \text{sets togther w/ a } G \text{ action} \\ G\text{-equivariant set function} \end{cases}$$

$$X \xrightarrow{f} Y$$

$$g \cdot_Y f(y) = f(g \cdot_G y)$$

thm $G\text{-}\mathbf{Set} \cong \mathbf{Set}^G$

$$\begin{tikzcd} G\text{-}Set \arrow[r, bend right, "f"'] & Set^G \arrow[l, bend right] \end{tikzcd}$$

Equivalence of categories

$$g f \cong 1_{G\text{-}Set}$$

$$f g \cong 1_{Set^G}$$

$$G\text{-}Set \xrightarrow{F} Set^G$$

$$(X, \underbrace{G \times X \to X}_{\substack{G \xrightarrow{s} Per(X) \\ g \mapsto s_g}}) \longmapsto F(X) : G \longrightarrow Set$$

$$\bullet \longmapsto X$$

$$g \longmapsto s_g$$

$$\begin{tikzcd} X \arrow[d, "f"'] \\ Y \end{tikzcd}$$

$$\begin{tikzcd} G \arrow[r, bend left, "F(X)"] \arrow[r, bend right, "G(X)"'] \arrow[r, phantom, "\Downarrow \alpha_f"] & Set \end{tikzcd}$$

$$\begin{tikzcd} X \arrow[r, "g\cdot(-)"] \arrow[d, "f"'] & X \arrow[d, "f"] \\ Y \arrow[r, "g(-)"] & Y \end{tikzcd}$$

$$Set^G \longrightarrow G\text{-}Set$$

$$F \longmapsto F(\bullet)$$