

Limits

- Products
- Equalizers
- Pullbacks
- General definitions

Colimits

- Coproducts
- Coequalizers
- Pushouts
- General def

- X of sets
- X in a poset coincide w/ \wedge .
- Vect $\text{Ker}(f)$ is an equalizer.
-

Monos \triangleleft Epis

- Monos in Set
- Epis in Set.
- $\mathbb{Z} \hookrightarrow \mathbb{Q}$ (in Rings).

Thm

all limits

\Leftrightarrow

\prod + equalizer.

colimits

\Leftrightarrow

\coprod + coequalizers.

(with) terminal object + pullbacks.



Adjunctions

two approaches

Empirical / Historical

free constructions in universal algebra



Groups

"In retrospect / Conceptually"

Generalized universal morphisms

Galois Connections

Kan extensions

generalized univers.

Groups

"The free group on a set X "

$F(X)$

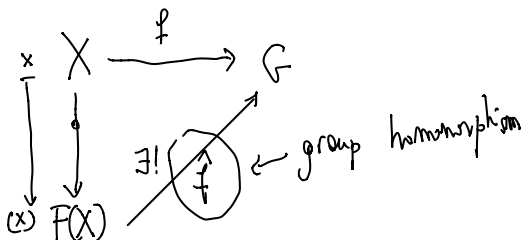
F_X

F_n



$\{ w_1 w_2 w_3 w_4 \dots, w_i^{-1}, w_i \in X \}$

$w_i^{-1} \cdot w_i = \text{empty word}$



$$\hat{f}(\alpha(x)) = f(x)$$

$$\alpha(x) \mapsto f(x) \cdot f(x) \dots$$

So there is a functor

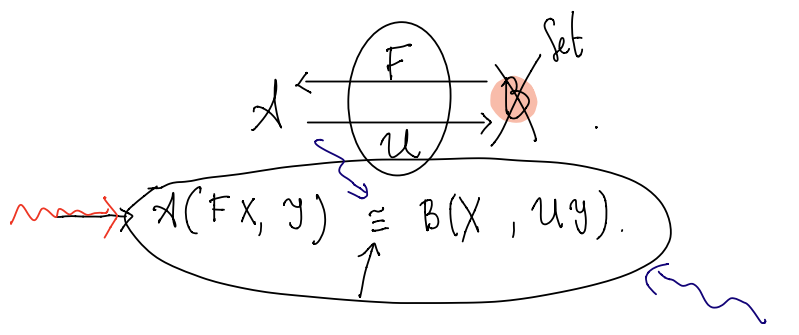
$$\begin{array}{ccc}
 \text{Set} & \xrightarrow{F} & \text{Grp} \\
 X & \xrightarrow{\quad} & F(X) \\
 f \downarrow & \rightsquigarrow & \uparrow \downarrow \\
 Y & & F(Y)
 \end{array}$$

$$\begin{array}{ccc}
 \text{Grp} & \xrightarrow{u} & \text{Set} \\
 G & \xrightarrow{\quad} & \{G\}
 \end{array}$$

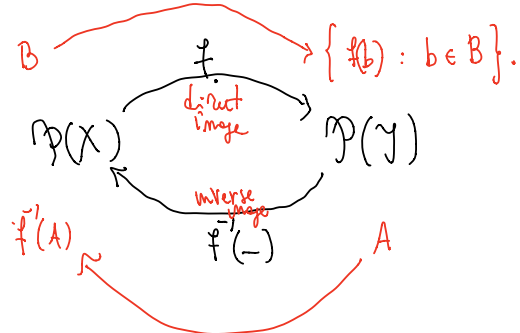
the universal property of F

$$\text{Grp}(F(X), G) \cong \text{Set}(X, u(G))$$

$\uparrow \hat{F} \quad \leftarrow f: X \rightarrow G$
↖ net function



$f: X \rightarrow Y$ is a set function



$\mathcal{P}(X)$ is a poset.
 $\mathcal{P}(X)$ is a category - $A \rightarrow B \Leftrightarrow A \subseteq B$.

f and f^{-1} are functors
 $A \subseteq B \Rightarrow f(A) \subseteq f(B)$.

Remark

$$\mathcal{P}(X)(f^{-1}A) \supseteq B \Leftrightarrow A \supseteq f(B)$$

$$\rightsquigarrow \mathcal{P}(X)(A, f^{-1}B) \cong \mathcal{P}(Y)(f(A), B).$$

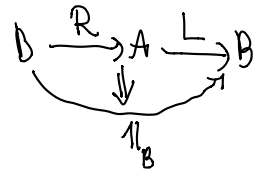
Grp

\mathbb{F}^1

Def Let $L: \mathcal{A} \rightleftarrows \mathcal{B}: R$ be functors between categories. We say that L is left adjoint to R (R is right adjoint to L) ($L \dashv R$) if there exist

$$\eta: \mathbb{1}_{\mathcal{A}} \Rightarrow RL$$

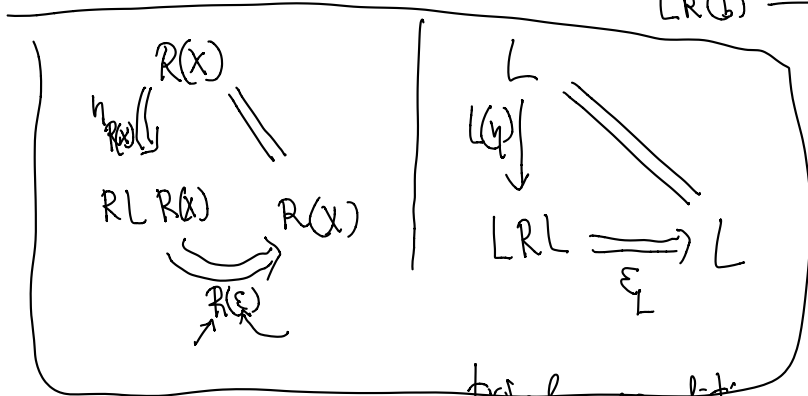
$$\epsilon: LR \Rightarrow \mathbb{1}_{\mathcal{B}}$$



such that

$\forall b \in \mathcal{B}$

$$LR(b) \xrightarrow{\epsilon_b} b \xrightarrow{\eta_b} LR(b)$$



① In the case of groups

$$\mathcal{F} \rightarrow \mathcal{U}$$

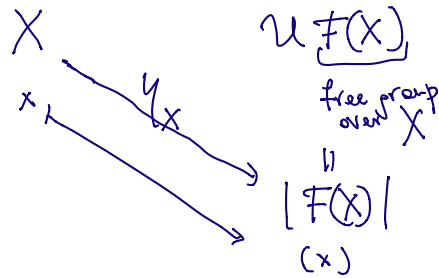
→ ② We try to recover the hom-set definition from the definition above.

1! $\mathcal{F}: \text{Set} \rightleftarrows \text{Grp}: \mathcal{U}$

$$\eta: X \rightarrow \mathcal{U}(X)$$

$$\epsilon: \mathcal{U}(X) \rightarrow X$$

"maps enter in" the right adjoint



group homomorphism

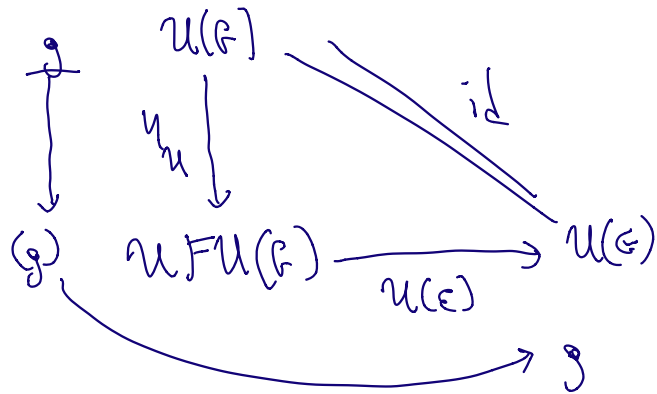
$$\epsilon: F(G) \rightarrow G$$

seen as a set

$$\{e, g_1, g_2, g_3, g_4\} \rightarrow \{g_1, g_2, g_3, g_4\}$$

generation of the group.

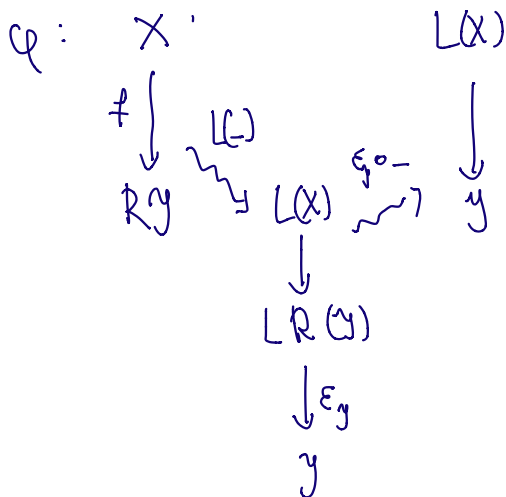
We check one triangle equality



Then $L: A \rightleftarrows B: R$, then

$$\varphi: \underline{A(X, RY)} \xrightarrow{\varphi^{-1}} B(LX, Y)$$

Proof



$$\varphi: \mathcal{A}(X, \mathcal{R}Y) \longrightarrow \mathcal{B}(LX, Y)$$

$$f \longmapsto \varepsilon_Y \circ L(f)$$

$$\varphi^{-1}: \mathcal{B}(L(X), Y) \longrightarrow \mathcal{A}(X, \mathcal{R}(Y))$$

$$g \longmapsto R(g) \cdot \eta_X$$

$$\begin{array}{ccccc}
 LX & & X & & X \\
 g \downarrow & & \downarrow \eta_X & & \downarrow \\
 Y & \xrightarrow{R(g)} & RL(X) & \xrightarrow{R(g) \circ \eta_X} & RY \\
 & & R(g) \downarrow & & \\
 & & RY & &
 \end{array}$$

$$\varphi^{-1} \varphi(f) = f.$$

$$\begin{aligned}
 \varphi^{-1}(\varepsilon_Y \circ L(f)) &= R(\varepsilon_Y \circ L(f)) \cdot \eta_X \\
 &= \underbrace{R(\varepsilon_Y) \circ L(f)}_{\text{triangle equality}} \cdot \eta_X = f.
 \end{aligned}$$

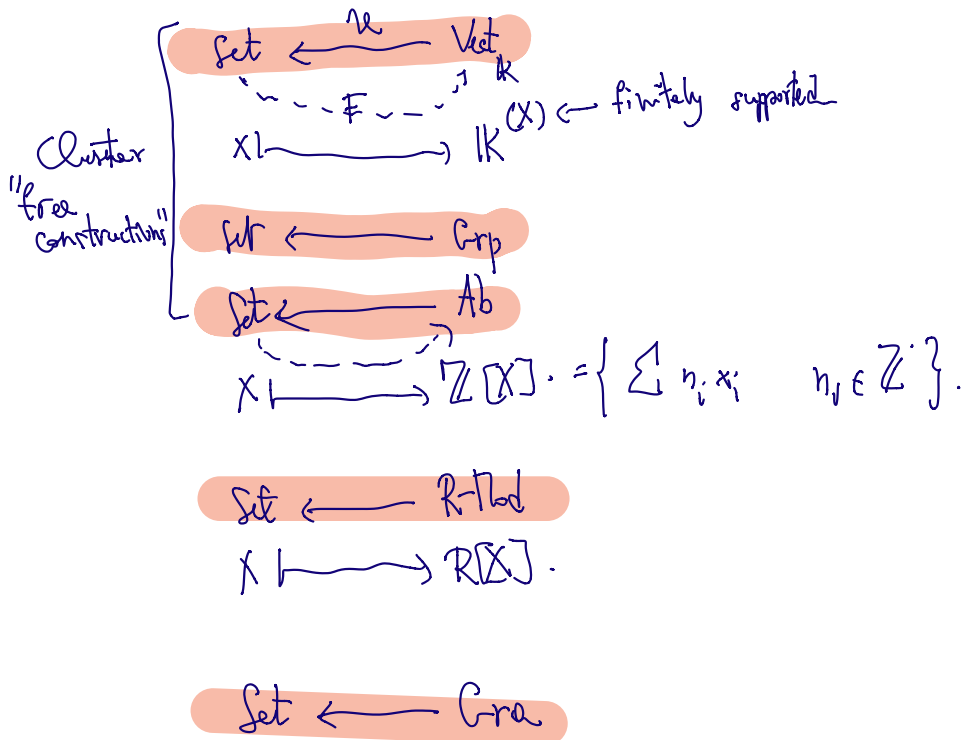
More is true the bijection is natural on X and Y

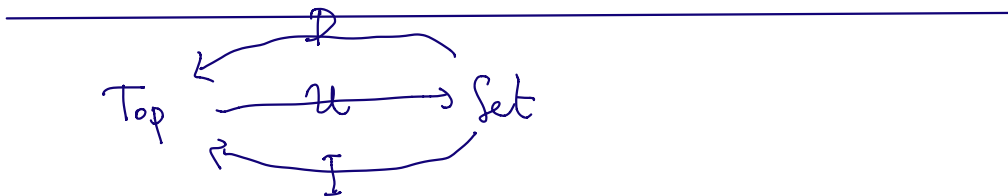
Adjunctions $(L, R, \eta, \varepsilon)$.

$$\{ [L, -] \cong [-, R] \}$$

1 example Groups

Examples.





$$\boxed{D \dashv u \dashv I.}$$

$$\text{Top}(D(X), Y) \cong \text{Set}(X, \underline{u(Y)})$$

$\uparrow f \quad \longleftarrow \quad \uparrow f$

$$\text{Set}(u(X), Y) \cong \text{Top}(X, I(Y))$$

$f \dashv \uparrow \longrightarrow f.$

Cartesian closed categories.

2, 5, 7 numbers

$$2^{(5 \times 7)} = (2^5)^7$$

$$\rightsquigarrow \text{Set}(5 \times 7, 2) \cong \text{Set}(7, \text{Set}(5, 2))$$

$$\rightsquigarrow \text{Set}(X \times Y, Z) \cong \text{Set}(Y, \text{Set}(X, Z))$$

$$\begin{array}{ccc}
 \text{Set} & \xrightarrow{X_x(-)} & \text{Set} \\
 y & \longmapsto & X_x y \\
 \\
 \text{Set} & \xleftarrow{(-)^x} & \text{Set} \\
 \mathbb{Z}^x & \longleftarrow & \mathbb{Z}
 \end{array}$$

In the category of sets $X_x(-) \dashv (-)^x$.

Def \uparrow cartesian closed
 $\forall X$ has right adj.

Who is the counit in this case?

$$\begin{array}{ccc}
 A \times B^A & \xrightarrow{\epsilon_B} & B \\
 \hline
 (a, f) & \longmapsto & f(a)
 \end{array}$$

evolution,

Notice that

Vect is not cartesian closed!! ($A \times B = \underline{A \oplus B}$)

$$(\text{Vect} \otimes) V \otimes _ : \text{Vect} \longrightarrow \text{Vect}$$

$$W \longmapsto V \otimes W.$$

$\forall V$ has a right adjoint!

$$\text{Vect} \longleftarrow \text{Vect} : [V, _]$$

$$W^V \longleftarrow W$$

$$\text{Vect}(A \otimes B, C) \cong \text{Vect}(A, C^B).$$

Monoidal closed
category

Ex not every forgetful functor has a left adjoint

$$\text{Set} \xleftarrow{u} \text{Fld} \xrightarrow{f} \text{Set}$$

(Note: Fld is circled in the original image)

$$\text{Set} \xleftarrow{u} \text{Set} \xrightarrow{L} \text{Set}$$

$$X \xleftarrow{u} (X, \otimes) \xrightarrow{f} (y, y_2)$$

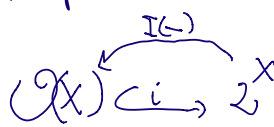
$f(x) = y_2$

is left adjoint to u .

$$(L : X \longmapsto (X \perp \{1\}, \uparrow))$$

Interior operators

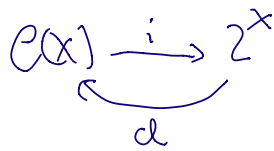
\mathcal{O} is a topology on X



i is a point function / functor

$i \rightarrow I(-)$

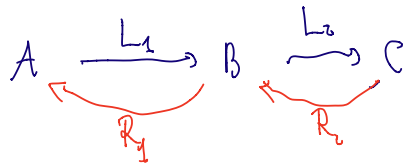
$$iU \ll X \Leftrightarrow U \leq I(X)$$



$d \rightarrow i.$

Prop

adjoints compose.



$L_2 \circ L_1$ has a right adjoint and it is $R_1 \circ R_2$

Proof

$$A(-, R_1 R_2 -) \cong B(L_1 -, R_2 -) \cong C(L_2 L_1 -, -)$$

