Limits	Colimits
· Products	· Coproducts
· Equalizers	· Coequalizers
· Pullbacks	· Pushouts
· General definitions	· General def

- $\times$ of sets
- $\times$ in a poset coincide w/ $\wedge$.
- Vect $\mathrm{Ker}(f)$ is an equalizer.
-

Monos & Epis

- Monos in Set
- Epis in Set.
- $\mathbb{Z} \hookrightarrow \mathbb{Q}$ (in Rings).

Thm all limits $\Leftrightarrow$ $\prod$ + equalizer.

colimits $\Leftrightarrow$ $\coprod$ + coequalizers.

terminal object + (with) pullbacks.

Adjunctions

two approaches

Empirical / Historical	"In retrospect / Conceptually"
"free constructions" in universal algebra	Generalized universal arrows
Groups	Galois Connections
	Kan extensions
	generalized universal arrows.

Groups "the free group on a set X"

$F(X)$ $\quad F_X \quad F_n$

$\left\{ w_1 w_2 w_3 w_4 \cdots \quad w_i \in X \right\}$.

w_i^{-1}

$w_i^{-1} \cdot w =$ empty word.

$$\begin{array}{ccc} x & X & \xrightarrow{f} & G \\ \downarrow & \downarrow & \nearrow_{\exists!\, \hat{f}} & \\ (x) & F(X) & & \end{array}$$

$\exists!\ \hat{f}$ ← group homomorphism

$\hat{f}((x)) = f(x)$.

$(x_i) \longmapsto f(x_1) \cdot f(x_2) \cdots$

So there is a functor

$$\begin{tikzcd} \mathbf{Set} \arrow[rr, "F"] & & \mathbf{Grp} \\ X \arrow[rr, mapsto] \arrow[d, "f"'] & & F(X) \arrow[d, "\hat{f}"'] \\ Y \arrow[rr, phantom, "\rightsquigarrow"] & & F(Y) \end{tikzcd}$$

$$\begin{tikzcd} \mathbf{Grp} \arrow[rr, "\mathcal{U}"] & & \mathbf{Set} \\ G \arrow[rr, mapsto] & & \lvert G \rvert . \end{tikzcd}$$

the universal property of F

$$\mathbf{Grp}(F(X), G) \cong \underline{\mathbf{Set}(X, \mathcal{U}(G))}$$

$$\hat{f} \quad \longleftarrow\!\!\sim\!\sim\!\sim \quad f : X \longrightarrow G$$

set function

$$\begin{tikzcd} \mathcal{A} \arrow[r, bend left, leftarrow, "F"] \arrow[r, bend right, "\mathcal{U}"'] & \cancel{\mathcal{B}}\ \mathbf{Set} . \end{tikzcd}$$

$$\mathcal{A}(FX, Y) \cong \mathcal{B}(X, \mathcal{U}Y).$$

$f : X \longrightarrow Y$ is a set function

$$\begin{tikzcd} \mathcal{P}(X) \arrow[r, bend left, "f_\bullet"] & \mathcal{P}(Y) \arrow[l, bend left, "f^{-1}(-)"] \end{tikzcd}$$

direct image

inverse image

$B \mapsto \{ f(b) : b \in B \}$.

$f^{-1}(A) \mapsfrom A$

$\mathcal{P}(X)$ is a poset. $\subseteq$

$\mathcal{P}(X)$ is a category. $A \to B \iff A \subseteq B$.

$f_\bullet$ and f^{-1} are functors

$A \subseteq B \Rightarrow f_\bullet(A) \subseteq f_\bullet(B)$.

Rem

$f^{-1}A \supseteq B \iff A \supseteq f_\bullet B$

$\rightsquigarrow \mathcal{P}(X)(A, f^{-1}B) \cong \mathcal{P}(Y)(f_\bullet A, B)$.

Grp $\qquad\qquad$ f^{-1}

Def Let $L : \mathcal{A} \rightleftarrows \mathcal{B} : R$ be functors between categories. We say that L is left adjoint to R (R is right adjoint to F) ($L \dashv R$) if there exist

$$\eta : \mathbb{1}_{\mathcal{A}} \Longrightarrow RL$$

$$\varepsilon : LR \Longrightarrow \mathbb{1}_{B}$$

$$\begin{tikzcd} B \arrow[r, "R"] \arrow[rr, bend right, "\mathbb{1}_B"'] & A \arrow[r, "L"] \arrow[d, phantom, "\Downarrow"] & B \\ & {} & \end{tikzcd}$$

$\forall\, b \in B$

$$LR(b) \xrightarrow{\varepsilon_b} b = \mathbb{1}_B(b).$$

such that

$$\begin{tikzcd} R(X) \arrow[d, "\eta_{R(X)}"'] \arrow[dr, equal] & \\ RLR(X) \arrow[r, "R(\varepsilon)"'] & R(X) \end{tikzcd} \qquad \begin{tikzcd} L \arrow[d, "L(\eta)"'] \arrow[dr, equal] & \\ LRL \arrow[r, "\varepsilon_L"'] & L \end{tikzcd}$$

triangle equalities

① In the case of groups

$$F \dashv U.$$

② We try to recover the hom-set ~~definition~~ from the definition above.

1! $F : \mathbf{Set} \rightleftarrows \mathbf{Grp} : U$

$\rightsquigarrow$ $\eta_X : X \longrightarrow UL(X)$ "maps enter in the right adjoint."

$\varepsilon_X : LU(X) \longrightarrow X$

$$\begin{array}{ccc} X & & U\underbrace{F(X)}_{\text{free group over } X} \\ & \searrow^{\eta_X} & \| \\ x & \longmapsto & |F(X)| \\ & & (x) \end{array}$$

group homomorphism

$\varepsilon_X : F(G) \longrightarrow G$

seen as a set

$\underbrace{(g_1)(g_2)(g_3)(g_4)} \longmapsto g_1 \cdot g_2 \cdot g_3 \cdot g_4$

· generation of the group.

We check <u>one</u> triangle equality

$$\begin{tikzcd}
g \arrow[d, mapsto] & \mathcal{U}(G) \arrow[d, "\eta_{\mathcal{U}}"'] \arrow[dr, bend left, "id"] & \\
(g) \arrow[drr, bend right] & \mathcal{U}F\mathcal{U}(G) \arrow[r, "\mathcal{U}(\varepsilon)"'] & \mathcal{U}(G) \\
& & g
\end{tikzcd}$$

<u>Thm</u> $L : \mathcal{A} \rightleftarrows \mathcal{B} : R$, then

$$\underbrace{\varphi : \mathcal{A}(X, RY) \rightleftarrows \mathcal{B}(LX, Y) : \varphi^{-1}}$$

<u>Proof</u>

$$\begin{tikzcd}
\varphi : & X \arrow[d, "f"'] & & & L(X) \arrow[d] \\
& RY \arrow[r, squiggly, "L(-)"] & L(X) \arrow[d] \arrow[rr, squiggly, "\varepsilon \circ -"] & & Y \\
& & LR(Y) \arrow[d, "\varepsilon_Y"] & & \\
& & Y & &
\end{tikzcd}$$

$$\varphi : \mathcal{A}(X, RY) \rightsquigarrow \mathcal{B}(LX, Y)$$

$$f \longmapsto \varepsilon_Y \circ L(f)$$

$$\varphi^{-1} : \mathcal{B}(L(X), Y) \rightsquigarrow \mathcal{A}(X, R(Y))$$

$$g \longmapsto R(g) \circ \eta_X .$$

$$\begin{tikzcd}
LX \arrow[d, "g"] & & X \arrow[d, "\eta_X"] & & X \arrow[d] \\
Y \arrow[rr, phantom, "\xrightarrow{R(-)}"] & & RL(X) \arrow[d, "R(g)"] \arrow[rr, phantom, "\xrightarrow{R(-)\circ\eta_X}"] & & RY \\
& & RY & &
\end{tikzcd}$$

$$\varphi^{-1}\underbrace{\varphi(f)}_{\shortparallel} = f.$$

$$\varphi^{-1}(\varepsilon_Y \circ L(f)) = R(\varepsilon_Y \circ L(f)) \circ \eta_X$$

$$= \underbrace{R(\varepsilon_Y) \circ L(f) \circ \eta_X}_{\text{triangle equality}} \quad = 1 .$$

More is true: the bijection is natural in $\underline{X}$ and $\underline{Y}$

Adjunctions $(L, R, \eta, \varepsilon)$

$$\Big\updownarrow$$

$$[L-, -] \cong [-, R-]$$

1 example <u>Groups</u>

Examples.

Cluster "free constructions"

$$\begin{tikzcd} \mathbf{Set} \arrow[r, dashed, bend right, "F"'] & \mathbf{Vect}_{\mathbb{K}} \arrow[l, "U"'] \end{tikzcd}$$

$X \longmapsto \mathbb{K}^{(X)}$ ← finitely supported

$\mathbf{Set} \longleftarrow \mathbf{Grp}$

$$\begin{tikzcd} \mathbf{Set} \arrow[r, dashed, bend right] & \mathbf{Ab} \arrow[l] \end{tikzcd}$$

$X \longmapsto \mathbb{Z}[X] := \{ \sum_i n_i x_i \quad n_i \in \mathbb{Z} \}$.

$\mathbf{Set} \longleftarrow R\text{-}\mathbf{Mod}$

$X \longmapsto R[X]$.

$\mathbf{Set} \longleftarrow \mathbf{Gra}$

$$
\begin{tikzcd}
\mathbf{Top} \arrow[r, "\mathcal{U}"] & \mathbf{Set} \arrow[l, bend right, "D"'] \arrow[l, bend left, "I"]
\end{tikzcd}
$$

$$\boxed{D \dashv \mathcal{U} \dashv I.}$$

$$\mathbf{Top}(D(X), Y) \cong \mathbf{Set}(X, \mathcal{U}(Y))$$

$$f \longleftarrow\!\shortmid\; f$$

$$\mathbf{Set}(\mathcal{U}(X), Y) \cong \mathbf{Top}(X, I(Y))$$

$$f \longmapsto f.$$

Cartesian closed categories.

2, 5, 7 numbers

$$2^{(5\times 7)} = \left(2^{5}\right)^{7}.$$

$$\|$$

$$\mathbf{Set}(5\times 7, 2) \cong \mathbf{Set}(7, \mathbf{Set}(5, 2))$$

$$\rightsquigarrow \quad \mathbf{Set}(X \times Y, Z) \cong \mathbf{Set}(Y, \mathbf{Set}(X, Z))$$

$$\mathrm{Set} \xrightarrow{X\times(-)} \mathrm{Set}$$
$$y \longmapsto X\times y$$

$$\mathrm{Set} \xleftarrow{(-)^X} \mathrm{Set}$$
$$Z^X \longleftarrow\!\shortmid\; Z$$

In the category of sets $X \times (-) \dashv (-)^X$.

Def $\forall X$ | cartesian closed: $X\times(-)$ has right adj.

Who is the counit in this case? evaluation,

$$A \times B^A \xrightarrow{\varepsilon_B} B$$
$$(a, f) \longmapsto f(a)$$

Notice that

Vect is not cartesian closed!! $(A \times B = A \oplus B)$

$(\mathrm{Vect}, \otimes)$ $\quad V \otimes - : \mathrm{Vect} \longrightarrow \mathrm{Vect}$

$$W \longmapsto V \otimes W.$$

$\forall V$ has a right adjoint!

$$\mathrm{Vect} \longleftarrow \mathrm{Vect} : [V, -]$$

$$W^V \longleftarrow\!\shortmid\; W$$

$$\mathrm{Vect}(A \otimes B, C) \cong \mathrm{Vect}(A, C^B).$$

Monoidal closed categories

Ex not every forgetful functor has a left adjoint

$$\begin{tikzcd}
\mathrm{Set} \arrow[r, dashed, bend left, "\cancel{F}"] & \mathrm{Fld} \arrow[l, "U"]
\end{tikzcd}$$

$$\begin{tikzcd}
\mathrm{Set} \arrow[r, bend left, "L"] & \mathrm{Set}_\bullet \arrow[l, "U"]
\end{tikzcd}$$

$$X \longmapsto (X, x_0) \xrightarrow{f} (Y, y_0)$$

$$f(x_0) = y_0.$$

is left adjoint to U.

$$L : X \longmapsto (X \amalg \{1\}, 1).$$

Interior operators

$\mathcal{O}$ is a topology on X

$$\begin{tikzcd} \mathcal{O}(X) \arrow[r, hook, "i"] & 2^X \arrow[l, bend right, "I(-)"'] \end{tikzcd}$$

i is a poset function / functor

$i \dashv I(-)$

$iU \leq X \iff U \leq I(X)$

$$\begin{tikzcd} \mathcal{C}(X) \arrow[r, "i"] & 2^X \arrow[l, bend left, "cl"] \end{tikzcd}$$

$cl \dashv i$.

Prop adjoints compose.

$$\begin{tikzcd} A \arrow[r, "L_1"] & B \arrow[l, bend left, "R_1"] \arrow[r, "L_2"] & C \arrow[l, bend left, "R_2"] \end{tikzcd}$$

$L_2 \circ L_1$ has a right adjoint and it is $R_1 \circ R_2$

Proof

$$A(-, R_1 R_2 -) \cong B(L_1 -, R_2 -) \cong C(L_2 L_1 -, -)$$