ON THE CANTOR-YONEDA EMBEDDING

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> Perhaps the purpose of categorical algebra is to show that which is trivial is trivially trivial.
>
> — J.P. Freyd

1. Definitions, basic constructions and properties

1.1. **Characters.** Let us introduce the main character of this play: posets and morphisms between them. The intense study of their property and interactions we be our toy model for category theory.

Definition 1.1.1 (Category)**.** A poset $\mathsf{C} = (C, \leq)$ is a set C equipped with a reflexive, antisymmetric and transitive relation.

Remark 1.1.2 (A dip in the analogy)**.** Since this is the first time that we use the analogy between posets and categories on which this chapter is built, let us remind what is the connection between the two structures. The elements of the posets should look like objects of a category, while the relation should carry the data about the arrows, for example the compositionality of the arrows is encoded in the transitivity of the relation. The experienced reader knows that this analogy can lead to the observation that every poset is a category in a natural way, built making formal the recipe we just described, the subtle abuse of this chapter will be to push the reader in treating any category as it was a poset.

Definition 1.1.3 (Functor)**.** A morphism of posets $f : \mathsf{C} \to \mathsf{D}$ is a set function preserving the poset structure. This means that if $a \leq b$, then $f(a) \leq f(b)$.

1.2. **Constructions.** In order to proceed with the next sections, we need to collect and perform some constructions with posets. This subsection is dedicated to exploring these basic constructions. This is far from being a complete exposition of all the possible constructions that can be performed with posets, it's just a list of those that we need and we find significant enough.

Definition 1.2.1 (Opposite category)**.** Given a poset C one can always define the opposite poset C°. The elements of C° are the same of the elements of C, while the order relation is defined by the following rule $c \leq_{\mathsf{C}^\circ} d$ if and only if $d \leq_{\mathsf{C}} c$.

Remark 1.2.2. Observe that even if the opposite C° has the same elements, it is incredibly far from being the same posets, in fact self dual posets are incredibly rare. Could you name one?

Definition 1.2.3 (Product of categories)**.** Given two posets C, D one can define the product of posets $\mathsf{C} \times \mathsf{D}$ as the cartesian product of the underlying sets $C \times D$ equipped with the *pointwise* order relation induced by C and D.

Remark 1.2.4 (**Cat** is complete and cocomplete)**.** For the reader which is acquainted with a tiny bit of category theory already, the product $\mathsf{C} \times \mathsf{D}$ constructed in this way has the universal property of the product in the category of posets, moreover many limits and colimits can be constructed in a similar manner. This would amount to show that the category of posets is complete and cocomplete. Since it is not the aim of this note, especially to check this property in detail, the reader can check this on his/her own or consult any standard reference.

Definition 1.2.5 (Functor categories)**.** Given two posets C, D, the set of morphisms of posets $\mathsf{Pos}(\mathsf{C}, \mathsf{D})$ admits a natural structure of poset which is given by the codomain.

$$f \leq g \text{ iff for all } c \in \mathsf{C}, f(c) \leq g(c).$$

This means that $\mathsf{Pos}(\mathsf{C}, \mathsf{D})$ is itself an object of the category of posets. This spots a very relevant feature of the *internal logic* of the category of posets, we will see how this will play a central role later. We will address to this set, equipped with this structure as the **internal hom**.

Remark 1.2.6 (Two words on the internal logic)**.** At this stage it is not easy to provide a definition for the idiom *internal logic.* In a nutshell the internal logic of a category is the family of those external constructions that can be performed inside the category. An *external construction* is often incarnated by a presheaf, thus one could say that the construction can be internalized if the presheaf is representable, i.e. there exists an object expressing the universal property of the construction. Limits, colimits, subobject classifiers, internal homs, are instances of this pattern. Let us try to motivate at least with an example the latter statement, given a family of objects c_i in a category C, the universal property of the product $\prod_i c_i$ is syintetized in the following equation:

$$\mathsf{C}(a, \prod_i c_i) \cong \prod_i \mathsf{C}(a, c_i),$$

this means that the object $\prod_i c_i$ represents internally the external construction encoded by the presheaf $\prod_i \mathsf{C}(-, c_i) \ : \ \mathsf{C}^\circ \longrightarrow \mathsf{Set}$. In this sense, for Set-enriched categories, presheaves provide the free *semantics* for universals.

1.3. Properties.

Remark 1.3.1 (Cartesian closedness)**.** It is absolutely well known that in basic aritmetics the following equation holds,

$$5^{(2\times 3)} = (5^2)^3$$

Recalling that 5^2 is the cardinality of the set of functions $\mathsf{Set}(2,5)$, one can recover the previous equation from a very important property of the category of sets, namely that there is an isomorphism (bijection),

$$\mathsf{Set}(\mathsf{A}\times\mathsf{B},\mathsf{C}) \overset{1}{\cong} \mathsf{Set}(\mathsf{A},\mathsf{Set}(\mathsf{B},\mathsf{C})).$$

In functional programming this phenomenon is called Currying, after Haskell Curry, while in categorical terms this is called cartesian closedness of the category of sets. The bijection is defined in a very simple way, it maps

$$f(\_,\_) \mapsto (a \mapsto f(a,\_)).$$

Being cartesian closed is not such a common property, for example the category of groups is not cartesian closed, idem for the category of topological spaces. In fact the possibility of having an internal notion of the *set of functions* show how expressive the internal logic of the category is. The category of posets Pos is *smart* enough to have this property, using the order defined in the Definition 1.2.5. This means that the following equation holds.

$$\mathsf{Pos}(\mathsf{C}\times\mathsf{D},\mathsf{E}) \cong \mathsf{Pos}(\mathsf{C},\mathsf{Pos}(\mathsf{D},\mathsf{E}))$$

The bijection is identical to the one provided in this remark for the category of sets and is in fact just a restriction of it. Moreover, it is not only a bijection of sets, it is also an isomorphism of posets.

2. Truth values and presheaf construction

In this section we introduce the poset of truth values $\mathbb{T}$. In our analogy this poset replaces the role that the category of sets has among categories,

$$\mathbb{T} : \mathsf{Pos} = \mathsf{Set} : \mathsf{Cat}.$$

We show that $\mathbb{T}$ somehow *controls* the whole category of posets in the same way that the category of sets plays a central role among categories. This exceptional property is encoded by a posetal version of the Yoneda embedding, which we will reduce to the standard representation set C its powerset 2^C via *bump functions.*

2.1. The enrichment.

Definition 2.1.1 (The category of sets)**.** We define the poset of truth values $\mathbb{T}$ to be the poset with two elements and the only possible non trivial inequality between the two,

$$\mathbb{T} := \{0 < 1\}.$$

[1]This shows that the presheaf $\mathsf{Set}(-\times\mathsf{B},\mathsf{C}) : \mathbf{Set}^{\circ} \longrightarrow \mathbf{Set}$ is indeed representable and coincides with $\mathsf{Set}(-,\mathsf{C}^{\mathsf{B}})$.

Remark 2.1.2 (Truth values and $\mathbf{2}$)**.** Among logicians, this is known as the poset of *truth values*, because 0 can be identified with false and 1 can be identified with true and plays an evident role in the theory boolean algebras. It could also be identified with the Sierpinski space. A possible shortname for the same object might also be 2, which is the cardinality of its underlying set, this probably is the best intuition for this short note and we recommend to maintain it. We avoid this notation because it would be indistinguishable from the poset with two elements and no relation between them.

Remark 2.1.3 (Categories are enriched over sets)**.** Given a poset C, recall that its posetal relation $(\leq)$ is by definition a subset of the cartesian product $\leq \subset C \times C$. It is well known that, in the category of sets, subsets are classified by functions into the set with two elements,

$$\chi_{(-)} : \mathsf{Sub}(C \times C) \leftrightarrows \mathsf{Set}(C \times C,\ 2) : (-)^{-1}(1).$$

The correspondence maps a subset $S \subset C \times C$ into its characteristic function χ_S, while in the opposite direction maps a function to the counter image of 1. As a result of this construction, we can study the characteristic function associated to the (partial) order relation $\langle -, - \rangle := \chi_{\leq}$.

$$\begin{aligned} \langle -, - \rangle : C \times C &\longrightarrow 2 \\ (c, d) &\mapsto \begin{cases} 1 \text{ if } c \leq d \\ 0 \text{ if not.} \end{cases} \end{aligned}$$

This parining is just another way to encode the partial oder and can be transformed into a morphism of poset by equipping 2 with the correct structure of poset,

$$\langle -, - \rangle : \mathsf{C}^{\circ} \times \mathsf{C} \longrightarrow \mathbb{T}.$$

Notice that the opposite on the left side of the product is needed in order to make this function into a morphism of posets, otherwise it would not be, in fact the function inverts the order in the first component.

2.2. **The presheaf construction.**

Definition 2.2.1 (The presheaf construction)**.** Given a poset C we will call the poset[2] $\mathsf{Pos}(\mathsf{C}^{\circ}, \mathbb{T})$ the power poset of C and indicate it via the notation $\mathscr{P}(\mathsf{C})$.

Remark 2.2.2 (The Yoneda embedding)**.** Given a poset C, we can use the Currying phenomenon to find a morphism of posets from the natural pairing associated to the poset structure as indicated below

$$\begin{aligned} よ_{\mathsf{C}} : \mathsf{C} &\longrightarrow \mathscr{P}(\mathsf{C}) \\ c &\mapsto \langle -, c \rangle. \end{aligned}$$

Remark 2.2.3 (Bump functions)**.** Allowing ourselves to be sloppy and identifying for a moment $\mathscr{P}(C)$ with 2^C, the Yoneda embedding sends every element c to a kind of *bump function* $よc$, which is 0 anywhere but on the element c itself. Since we need to take into account the structure of poset, the bump function is mollified and is thus non-zero on a bigger set. The following lemma shows technically what

[2] Equipped with the order defined in Definition 1.2.5.

we have hinted in this remark, a morphism $f : \mathsf{C}^\circ \to \mathsf{T}$ does not vanish on an element c iff it is bigger than its associated bump function $\text{よ}c$.

Lemma 2.2.4 (Yoneda Lemma).

$$\text{よ}c \leq f \text{ iff } f(c) = 1$$

Proof. We organize the proof proving the two implications separately.

$\Rightarrow$) $\text{よ}c \leq f$ means that, for every $d \in \mathsf{C}$, when $\langle d, c \rangle = 1$ also $f(d) = 1$. Now, by definition $\langle c, c \rangle = 1$, because $c \leq c$, thus $f(c)$ must be 1.

$\Leftarrow$) Since $f(c) = 1$, for every element $d \leq c$, $f(d) = 1$, because f must (anti-)preserve the order relation of C. Since $\text{よ}c$ is 0 elsewhere, this is enough to show that $\text{よ}c \leq f$.

□

Remark 2.2.5 (Presheaves are complete and cocomplete). The poset $\mathcal{P}(\mathsf{C})$ inherits its supremum and meets[3] from $\mathbb{T}$, this means that given any family of functions $\mathcal{F} = \{f_i\}_{i \in I}$ one can define both the supremum and the infimum of the family in the following way.

$$(\sup \mathcal{F})(c) = \sup_{f_i \in \mathcal{F}} f_i(c).$$

Remark 2.2.6 (Representables are tiny). In the spirit of Rem. 2.2.3 and the Yoneda Lemma 2.2.4, we can observe that a bump function $\text{よ}c$ is an *atom* among the elements of $\mathcal{P}(\mathsf{C})$. To make evident its *undivisidibiliy*, imagine that $\text{よ}c = \sup f_i$, then we show that it must coincide with one of functions in the supremum. Since each f_i is smaller than the supremum, we already know that $f_i \leq \text{よ}c$, thus it is enough to show that there must be an i such that $\text{よ}c \leq f_i$. Obviously there must be an i such that $f_i(c) = 1$, but then by applying the Yoneda lemma 2.2.4, $\text{よ}c \leq f_i$, which is the thesis.

In the following corollary we observe that collecting the all the bump functions below f one can recover f itself. For graphical reasons, we indicate with the centerdot the meet.

Corollary 2.2.7 (Ninja Yoneda Lemma/Representables are dense among presheaves).

$$f = \sup_{c \in \mathsf{C}} (f \cdot \text{よ}c).$$

Proof. Using the Yoneda Lemma 2.2.4 its very simple to show that the two functions have the same values. □

Remark 2.2.8 (The correct notion of powerset). It is probably already evident to the reader, but let us stress the anology between the powerposet and the powerset. $\mathcal{P}(\mathsf{C})$ is the *correct* notion of 2^C in the category of posets. The rest of this subsection is dedicated to enhance the correspondence

$$\chi_{(-)} : \mathsf{Sub}(C) \leftrightarrows \mathsf{Set}(C, 2) : (-)^{-1}(1),$$

in the context of posets. In order to do so we need to introduce the correct notion of subset and study more closely our powerposet construction.

[3]Arbitrary suprema and suprema are interchangeable terms, idem for arbitrary meets and infima.

2.2.1. *Lower sets and the Cantor embedding.*

Definition 2.2.9 (Discrete opfibration)**.** Let C be a poset. An lower $\mathtt{I} \subset \mathsf{C}$ is an downward closed subset, i.e. if $i \in \mathsf{i}$ and $i \geq i'$, then $i' \in \mathtt{I}$. The set of lower sets $\mathsf{Low}(\mathsf{C})$ of a poset C is itself a poset ordered by inclusion.

Remark 2.2.10 (The Cantor embedding)**.** Given a poset C, we can construct a morphism of posets $\iota_{\mathsf{C}} : \mathsf{C} \longrightarrow \mathsf{Low}(\mathsf{C})$ which sends an elements c to the lower set (c) generated by c. This construction is evidently a variant of the trivial inclusion of a set in the set of its subsets, using singletons. This motivates the name that we have chosen.

2.2.2. *The Grothendieck construction.*

Remark 2.2.11. We have now introduced enough elements to picture the main diagram of this section and state our version of what's known under the name of *Grothendieck construction*. In the notation of diagram below,

$$\begin{tikzcd} & \mathsf{C} \arrow[dl, "よ_{\mathsf{C}}"'] \arrow[dr, "\iota_{\mathsf{C}}"] & \\ \mathscr{P}(\mathsf{C}) \arrow[rr, bend right, "(よ/-)"'] & & \mathsf{Low}(\mathsf{C}) \arrow[ll, bend right, "\chi_{(-)}"'] \end{tikzcd}$$

we will show that there exist that correspondence in the lower level of the diagram and is indeed an isomorphism of posets. Moreover, the diagram commute, this give us a conceptual representation for the Yoneda embedding, which coincides, up to isomorphism of posets to the Cantor embedding, associating to every element its naturally associated lower set,

$$\iota_C(-) = (よ/よ(-)).$$

Also, we fully recover our *bump function* intuition over the Yoneda embedding, in fact it can be built via the *characteristic function construction*,

$$よ_{\mathsf{C}}(-) = \chi_{\iota_{\mathsf{C}}(-)}.$$

Proposition 2.2.12 (The Grothendieck construction)**.** There exists an isomorphism of posets as indicated below.

$$\chi_{(-)} : \mathsf{Low}(\mathsf{C}) \leftrightarrows \mathscr{P}(\mathsf{C}) : (よ/-)$$

Proof. We split the proof in parts, we show the existence of the two morphism separately. The fact that they are one the inverse of the other is more or less a tautology.

$\chi_{(-)}$ maps a lower set $\mathtt{I}$ to the characteristic function $\chi_{\mathtt{I}}$,

$$c \mapsto \begin{cases} 1 \text{ if } c \in \mathtt{I} \\ 0 \text{ if not.} \end{cases}$$

$\chi_{\mathtt{I}}$ is a morphism of posets because $\mathtt{I}$ is a lower set, this prevents $\chi_{\mathtt{I}}$ to not preserve the poset relation.

($\text{よ}/-$) The reader has probably guessed the construction of ($\text{よ}/-$) and indeed the following is not the most concise way to present it, yet let us give this precise construction because this is the one that can be generalized to different contexts. Given an element $f \in \mathscr{P}(C)$ we can individuate it as a morphism $\lfloor f \rfloor : 1 \longrightarrow \mathscr{P}(C)$ which is *pointing towards* f. We might call $\lfloor f \rfloor$ the *name* of f. Now define $(\text{よ}/f)$ as a of *lax* pullback[4] of the following diagram.

$$(\text{よ}/f) := \{c \in \mathsf{C} : \text{よ}(c) \leq f\}^5$$

$$\begin{tikzcd}
(\text{よ}/f) \arrow[r, dashed] \arrow[d, dashed] & \mathsf{C} \arrow[d, "\text{よ}"] \\
1 \arrow[r, "\lfloor f \rfloor"] & \mathscr{P}(\mathsf{C})
\end{tikzcd}$$

By the Yoneda lemma 2.2.4 $(\text{よ}/f)$ coincides with $f^{-1}(1)$, which is probably the construction that the reader has guessed at the beginning of the proof.

□

2.2.3. *Functoriality of the presheaf construction.*

Remark 2.2.13. Let's go back for one moment to sets and functions, where our intuition is solid and trustable. Given any (set) function $f : A \longrightarrow B$ there are two very naturally induced functions between their powersets.

$$\begin{tikzcd}
A \arrow[rr, "f"] \arrow[d, "\iota_A"] & & B \arrow[d, "\iota_B"] \\
\mathsf{Sub}(A) \arrow[rr, bend left, "f_!"] & & \mathsf{Sub}(B) \arrow[ll, bend left, "f^{-1}"]
\end{tikzcd}$$

$f_!$ is often called *direct image* and maps a subset $S \subset A$ to the the subset of its images
$$S \mapsto \bigcup_{s \in S} f(s).$$

f^{-1} is often called the *inverse image* and maps a subset $S \subset B$ to the the subset of its pre-images images
$$S \mapsto f^{-1}(S).$$

Both these maps have interesting descriptions also in terms of bump functions, in that case for example the inverse image correspondes to the precomposition with f, for this reason it is very often indicated with f^*.

Remark 2.2.14. It is not at all surprising that this construction generalizes trivially to the case of posets when subsets are replaced by lower sets. This leads us to the following diagram.

[4]This terminology must be intended in an informal sense, $(\text{よ}/f)$ is not the pullback of that diagram, it is a very loose notion of pullback.

[5]Notice that the pullback would have been $(\text{よ}/f) := \{c \in \mathsf{C} : \text{よ}(c) = f\}$. That's the sense in which the pullback is only lax.

$$\begin{tikzcd}
\mathsf{C} \arrow[rr, "f"] \arrow[d, "\iota_{\mathsf{C}}"] \arrow[dd, bend right=60, "\text{よ}_{\mathsf{C}}"'] & & \mathsf{D} \arrow[d, "\iota_{\mathsf{D}}"] \arrow[dd, bend left=60, "\text{よ}_{\mathsf{D}}"] \\
\mathsf{Low}(\mathsf{C}) \arrow[rr, bend right, "f_!"'] \arrow[d, "\chi_{(-)}"] & & \mathsf{Low}(\mathsf{D}) \arrow[ll, bend right, "f^{-1}"'] \arrow[d, "\chi_{(-)}"] \\
\mathscr{P}(\mathsf{C}) \arrow[rr, bend right, dashed, "\mathscr{P}_! f"'] & & \mathscr{P}(\mathsf{D}) \arrow[ll, bend right, dashed, "\mathscr{P}^* f"']
\end{tikzcd}$$

The dotted maps are thus defined by composition in the only possible way provided the elements that we have,

$$\mathscr{P}_! f = \chi_{(-)} \circ f_! \circ (\text{よ}/-).$$

$$\mathscr{P}^* f = (\text{よ}/-) \circ f^{-1} \circ \chi_{(-)}.$$

We advice the reader to maintein the intuition provided by the action of these constructions over the lower sets and keep in mind that $\mathscr{P}^* f$ has a simple description in terms of precomposition.

$$\mathscr{P}^* f(-) = (-)\circ f.$$

We encourage the reader to check explicitely that these two definition of $\mathscr{P}^* f$ are in fact completely equivalent, this exercise is perfect to built the intuition.

Remark 2.2.15 (The Yoneda structure on **Cat**)**.** Going back to our prolegomena to the functoriality of the presheaf construction, Rem. 2.2.13, Given any (set) function $f : A \longrightarrow B$ there are two very naturally induced functions between their powersets.

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "\iota_A"] & B \arrow[d, "\iota_B"] \\
\mathsf{Sub}(A) \arrow[r, bend left, "f_!"] & \mathsf{Sub}(B) \arrow[l, bend left, "f^{-1}"]
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "\iota_A"] & B \arrow[dl, bend left, "f^{-1}"] \\
\mathsf{Sub}(A) &
\end{tikzcd}$$

In particular this provides us with a very weak notion of inverse $B \longrightarrow \mathsf{Sub}(A)$, as indicated on the right dide of the diagram above. Following Rem. 2.2.14, we can find a very similar construction also for posets.

$$
\begin{tikzcd}
\mathsf{C} \arrow[rr, "f"] \arrow[d, "\iota_{\mathsf{C}}"] \arrow[dd, bend right=60, "\text{よ}_{\mathsf{C}}"'] & & \mathsf{D} \arrow[d, "\iota_{\mathsf{D}}"] \arrow[dd, bend left=60, "\text{よ}_{\mathsf{D}}"] \\
\mathsf{Low}(\mathsf{C}) \arrow[d, "\chi(-)"] \arrow[rr, bend right=10, "f_*"'] & & \mathsf{Low}(\mathsf{D}) \arrow[d, "\chi(-)"] \arrow[ll, bend right=10, "f^{-1}"'] \\
\mathscr{P}(\mathsf{C}) \arrow[rr, dashed, bend right=10, "\mathscr{P}_*(f)"'] & & \mathscr{P}(\mathsf{D}) \arrow[ll, dashed, bend right=10, "\mathscr{P}^*f"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathsf{C} \arrow[r, "f"] \arrow[d, "\text{よ}_{\mathsf{C}}"] & \mathsf{D} \arrow[dl, dashed, bend left, "\langle f,-\rangle"] \\
\mathscr{P}(\mathsf{C}) &
\end{tikzcd}
$$

We define $\langle f, -\rangle$ to be the composition $\mathscr{P}^*f \circ \text{よ}_{\mathsf{C}}$, this definition is coherent with the chosen notation, as $\mathscr{P}^*f \circ \text{よ}(c)$ is by definition $\text{よ}\, c \circ f$.

2.3. **Cocomplete categories & algebras for the presheaf construction.** This subsection studies a very important relation between the existence of suprema and the powerposet construction. In particular we will show that a poset has suprema if and only if the Yoneda embedding is a retract,

$$\text{よ}_{\mathsf{C}} : \mathsf{C} \leftrightarrows \mathscr{P}(\mathsf{C}) : \mathsf{Int}_{\mathsf{C}}.$$

Remark 2.3.1. The notation $\mathsf{Int}_{\mathsf{C}}$ stands for *internalization* and was somehow motivated by the Rem. 1.2.6 in the case of presheaf categories. The same intuition would appear a bit artificial in the case of posets, thus we will not try to motivate it. The construction associates to a function $\mathsf{C}^\circ \to \mathbb{T}$ the *closest* approximation of f among bump functions.

Remark 2.3.2. Recall that C has suprema if, given any family of $c_i \in \mathsf{C}$, there exists a common upper bound $\sup c_i$, and moreover this upperbound is initial among upperbounds for the family. In a similar manner, one can define the notion of supremum of a morphism $f : \mathsf{D} \to \mathsf{C}$ and its very easy to say that a poset has suprema if and only if it has suprema for every morphism of posets.

Proposition 2.3.3 (Cocomplete categories are algebras for the presheaf construction)**.** If C has suprema, then $\text{よ} : \mathsf{C} \to \mathscr{P}(C)$ is a retract.

Proof. The proof is incredibly easy, observe that the inclusion of a lower set $\mathsf{I} \subset \mathsf{C}$ gives a morphism of posets, this means that when C has suprema there exists a morphism of posets

$$\sup : \mathsf{Low}(\mathsf{C}) \to \mathsf{C}.$$

Now we use this morphism to build the desired retraction as indicated by the diagram below.

$$
\begin{tikzcd}
\mathscr{P}(\mathsf{C}) \arrow[r, dashed, "\mathsf{Int}_{\mathsf{C}}"] \arrow[d, "(\text{よ}/-)"] & \mathsf{C} \\
\mathsf{Low}(\mathsf{C}) \arrow[ur, "\sup"'] &
\end{tikzcd}
$$

The Yoneda Lemma 2.2.4, together with the proof of Grothendieck construction 2.2.12 allows us to provide an explicit formula to cumputer this retraction.

$$\mathsf{Int}_{\mathsf{C}}(f) = \sup_{c \in f^{-1}(1)} c$$

Via this formula it is completely evident that the composition $\mathsf{Int}_{\mathsf{C}} \circ よ_{\mathsf{C}}$ equals the identity of C, as show by the following equation.

$$(\mathsf{Int}_{\mathsf{C}} \circ よ)(c) = \sup_{d \in よ c^{-1}(1)} d = c.$$

The last equality in the previous euqtion is true because c is by definition terminal (is the greatest) among those elements on which $よ c$ does not vanish. □

Remark 2.3.4 (A strategy for the converse)**.** At this point we would like to show also the other implication on the previous statement. In order to do so we proceed in two steps.

(1) *The comprehensive factorization system/Suprema of lower sets are enogh.* For every morphism $f : \mathsf{D} \to \mathsf{C}$, there exists a lower set $\mathtt{I}_f \subset \mathsf{C}$ and a factorization

$$\begin{tikzcd} \mathsf{D} \arrow[rr, "f"] \arrow[dr, "p_f"'] & & \mathsf{D} \\ & \mathtt{I}_f \arrow[ur, "i_f"'] & \end{tikzcd}$$

such that the sumpremum $\sup f$ exists if and only if $\sup i_f$ exists and in this case they coincide.

(2) if some retraction exists, then suprema of lower sets exist too.

Putting toghether this two steps one has shown that: a retraction exists $\overset{2}{\Longrightarrow}$ suprema of lower sets exist $\overset{1}{\Longrightarrow}$ suprema exist, and thus we have the converse of the previous theorem.

Proof of 2.3.4(1). This is relatively easy to check, define $\mathtt{I}_f$ to be the lower set generated by the image of f, since we are just adding elements below the image, the supremum does not change at all. □

Proof of 2.3.4(2). Assume that there exist some retraction $r : \mathscr{P}(\mathsf{C}) \to \mathsf{C}$. Now, using the Ninja Yoneda lemma 2.2.7, we have that

$$r = \sup_{d \in \mathsf{C}}(r \cdot よ c) = \sup_{d \in r(-)^{-1}(1)} よ c.$$

The last formula to compute r shows that r must preserve suprema. But if r preserves suprema, then its easy to show that it must coincide with the function computing the supremum of the associated lower set, thus suprema of lower sets exist. □

Corollary 2.3.5 (Algebras for the presheaf construction are cocomplete categories)**.**

Proof. Evident from Rem. 2.3.4. □

Proposition 2.3.6 (Universal property of the presheaf construction)**.** . The powerposet is *universal* among the completion under suprema. I.e. given any poset D with suprema, and a morphism of posets $\mathsf{C} \to \mathsf{D}$,

$$\begin{tikzcd} \mathsf{C} \arrow[r, "f"] \arrow[d, "よ_{\mathsf{C}}"'] & \mathsf{D} \\ \mathscr{P}(\mathsf{C}) \arrow[ur, dashed, "\mathrm{lan}_{よ} f"] & \end{tikzcd}$$

there exists a suprema-preserving extension $\mathrm{lan}_{よ} f : \mathscr{P}(\mathsf{C}) \to \mathsf{D}$ of f along the Yoneda embedding.

Proof. Putting together Prop. 2.3.3 and Rem. 2.2.14, the construction of $\mathrm{lan}_{よ} f$ is somehow the only possible one.

$$\begin{tikzcd} \mathsf{C} \arrow[r, "f"] \arrow[d, "よ_{\mathsf{C}}"'] & \mathsf{D} \\ \mathscr{P}(\mathsf{C}) \arrow[ur, dashed, "\mathsf{Int}_{\mathsf{D}} \circ \mathscr{P}_! f"] \arrow[r, "\mathscr{P}_! f"'] & \mathscr{P}(\mathsf{D}) \arrow[u, "\mathsf{Int}_{\mathsf{D}}"'] \end{tikzcd}$$

Now we need to show that $\mathrm{lan}_{よ} f$ defined in this way extends f along the Yoneda embedding, which means that we must check that $\mathsf{Int}_{\mathsf{D}} \circ \mathscr{P}_! f \circ よ_{\mathsf{C}} = f$, now this is just a matter of writing everything explicitely. It is easy to check from the definitions that $\mathscr{P}_! f \circ よ_{\mathsf{C}} = よ_{\mathsf{D}} \circ f$, and thus we get $\mathsf{Int}_{\mathsf{D}} \circ \mathscr{P}_! f \circ よ_{\mathsf{C}} = \mathsf{Int}_{\mathsf{D}} \circ よ_{\mathsf{D}} \circ f = f$. Finally the extension preserves suprema because both $\mathscr{P}_! f$ and $\mathsf{Int}_{\mathsf{D}}$ do so. □

Remark 2.3.7. We can give an operative description of the action of $\mathrm{lan}_{よ} f$. Given an element $g \in \mathscr{P}(\mathsf{C})$, we write it using the Ninja Yoneda Lemma 2.2.7 as the supremum of its nontrivial bump functions $g = \sup_{c \in g^{-1}(1)} よ c$, then we define $\mathrm{lan}_{よ} f$ by supremum preserving extension

$$\mathrm{lan}_{よ} f(g) = \sup_{c \in g^{-1}(1)} f(c).$$

One can check that this is precisely what is happening in the proof above.