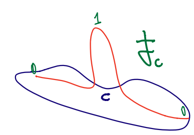
Last time

Posets have a natural pairing

$$\langle \, , \rangle \quad P^{o} \times P \longrightarrow \mathbb{I}$$

"Yoneda embedding"

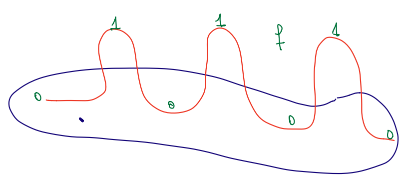
$$P \xrightarrow{\text{よ}} \mathbb{I}^{P^{o}}$$



Yoneda lemma

$$\text{よ}_c \leq f \iff f(c) = 1$$

Yoneda ninja / Representables are dense



$$f = \bigvee_{c \,:\, \text{よ}_c \leq f} \text{よ}_c$$

thm

$$\begin{tikzcd} P \arrow[r, "f"] \arrow[d, "\text{よ}_P"'] & Q \\ \mathbb{I}^{P^{o}} \arrow[ur, dotted, "\hat{f}"'] & \end{tikzcd}$$

$\forall f \ \forall Q$ with sups $\exists! \ \hat{f}$ sup-preserving :

$$f = \hat{f} \circ \text{よ}_P$$

thm P has sups iff

$$P \xrightarrow{\text{よ}} \mathbb{I}^{P}$$

has a retraction

Today. We discuss about "the same" in Categories.

① the hom pairing $(a,b) \longmapsto \mathcal{C}(a,b)$.

② the functoriality

$$\begin{tikzcd} b \arrow[d, "f"'] & & \mathcal{C}(a,b) \arrow[d, "f\circ -"] \\ b' & \longmapsto & \mathcal{C}(a,b') \end{tikzcd}$$

$$\begin{tikzcd} a \arrow[d, "g"'] & & \mathcal{C}(a,b) \\ a' & \longmapsto & \mathcal{C}(a',b) \arrow[u, "-\circ g"'] \end{tikzcd}$$

③ $\forall\, c \in \mathcal{C}$

representable functors

$$\mathcal{C}(c,-) : \mathcal{C} \longrightarrow \mathrm{Set}.$$
$$\mathcal{C}(-,c) : \mathcal{C}^{o} \longrightarrow \mathrm{Set}.$$

"bump function" over c

<u>Exa</u> $\mathrm{id} : \mathrm{Set} \longrightarrow \mathrm{Set}$.

$\mathrm{Top} \longrightarrow \mathrm{Set}$.

$\mathrm{Grp} \longrightarrow \mathrm{Set}$.

$\mathrm{hTop}_{\bullet} \longrightarrow \mathrm{Grp} \longrightarrow \mathrm{Set}$.

"forgetful are representable".

<u>Exa</u> $\mathrm{Set}^{op} \xrightarrow{\ \mathcal{P}\ } \mathrm{Set}$

$\mathrm{Top}^{op} \longrightarrow \mathrm{Set}$.

the hom-pairing.

$$\begin{aligned} \langle\, ,\rangle : \mathcal{C}^{op} \times \mathcal{C} &\longrightarrow \mathbf{Set} \\ (c\ ,\ c') &\longmapsto \mathcal{C}(c, c') \\ f\uparrow \quad g\downarrow &\longmapsto g \circ - \circ f \end{aligned}$$

the Yoneda embedding

$$\mathcal{C} \xrightarrow{\ \sharp\ } \mathbf{Set}^{\mathcal{C}^{op}}$$

What is this?

Yoneda Lemma

$$\mathbf{Set}^{\mathcal{C}^{op}}(\sharp_c, X) \cong X(c).$$

$$\sharp_c \leq X \iff X(c) = 1.$$

$$
\begin{aligned}
\mathrm{Set}^{\mathcal{C}^{op}}(\text{よ}_c, X) &\longrightarrow X(c) \\
\alpha : \text{よ}_c \Rightarrow X &\longmapsto \alpha_c(1_c) \\
\alpha_c : \mathcal{C}(c,c) &\longrightarrow X(c) \\
1_c &\longmapsto \alpha_c(1_c)
\end{aligned}
$$

$$
\begin{aligned}
X(c) &\longrightarrow \mathrm{Set}^{\mathcal{C}^{op}}(\text{よ}_c, X) \\
x &\longmapsto \bar{x} : \text{よ}_c \Rightarrow X \\
\bar{x}_a : \mathcal{C}(a,c) &\longrightarrow X(a) \\
f &\longmapsto X(f)(x)
\end{aligned}
$$

Sanity check.

$$
\begin{aligned}
X(c) \longrightarrow \mathrm{Set}^{\mathcal{C}^{op}}&(\text{よ}_c, X) \longrightarrow X(c) \\
x \longmapsto \quad &\bar{x} \longmapsto \bar{x}_c(1_c) \\
&\qquad\quad \| \\
&\qquad 1_c(x) = x.
\end{aligned}
$$

$\boxed{\text{thm. the Yoneda embedding is f.f.}}$

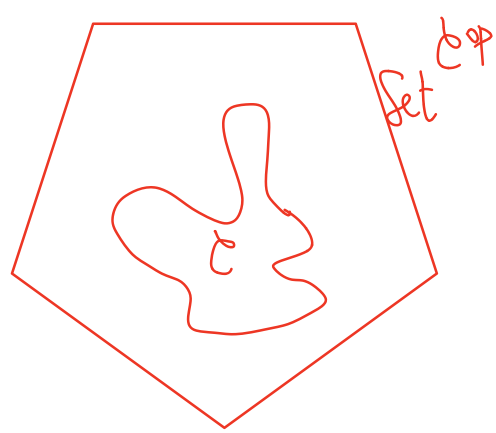
$\text{よ}: \mathcal{C} \longrightarrow \mathrm{Set}^{\mathcal{C}^{op}}.$

$$\text{よ}: \mathcal{C}(a,b) \longrightarrow \mathrm{Set}^{\mathcal{C}^{op}}(\text{よ}a, \text{よ}b)$$

$\uparrow$ bijective.

$\cong$ Yoneda lemma

$$\text{よ}_b(a) \cong \mathcal{C}(a,b).$$



<u>Cor</u> $a \simeq b$ iff $f_a \simeq f_b$.

The identity of Set is a rigid function.

CATEGORY THEORY

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EXERCISES

Leinster ( ). 4.1.11

Leinster ( ). 4.1.28

Leinster ( ). 4.1.32

Leinster ( ). 4.3.18

Riehl ( ). 2.1.ii

Riehl ( ). 2.2.v

Riehl ( ). 2.2.vii

- the exercises in the red group are mandatory.
- pick at least one exercise from each of the yellow groups.
- pick at least two exercises from each of the blue groups.
- nothing is mandatory in the brown box.
- The riddle of the week. It's just there to let you think about it. It is not a mandatory exercise, nor it counts for your evaluation. Yet, it has a lot to teach.
- useful to deepen your understanding. Take your time to solve it. (May not be challenging at all.)
- measures the difficulty of the exercise. Note that a technically easy exercise is still very important for the foundations of your knowledge.
- It's just too hard.

The label **Leinster** refers to the book **Basic Category Theory**, by *Leinster*.
The label **Riehl** refers to the book **Category Theory in context**, by *Riehl*.

Date: November 3, 2020.