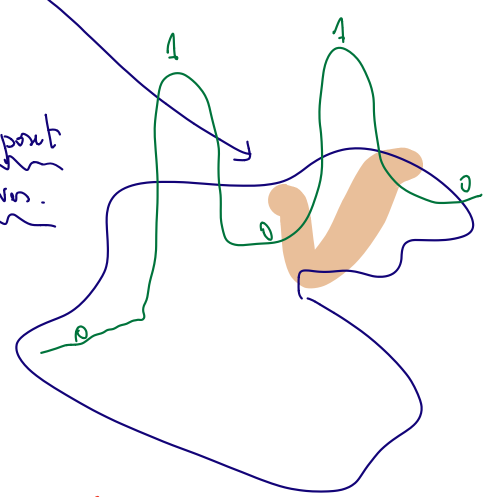
Last time Baby Yoneda lemma (poset).

$$\langle -,- \rangle : P^{o} \times P \longrightarrow \Pi$$

$$(p, q) \longmapsto \begin{cases} 1 & p \le q \\ 0 & \text{otherwise.} \end{cases}$$

pairing

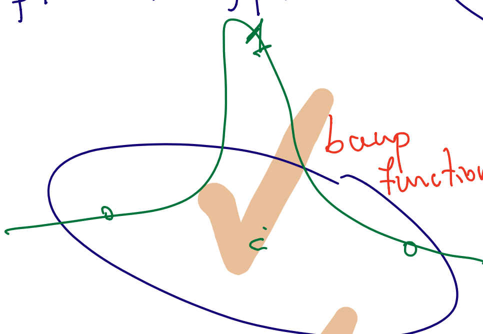


Yoneda embedding

$$P \xrightarrow{\;y\;} (\Pi^{P^{op}} \ni f)$$

power poset
presheaves.

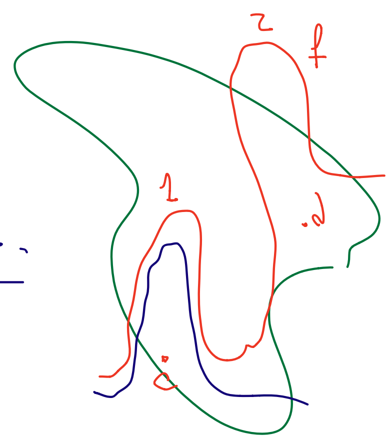
$$p \longmapsto \langle -, p \rangle .$$



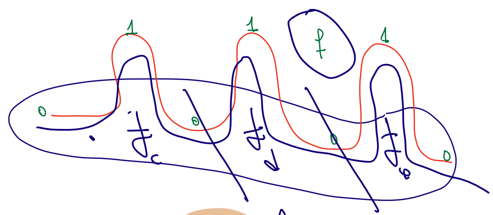
bump function/functional

Yoneda lemma

$$y_c \le f \iff f(c) = 1 .$$



Yoneda ninja / Representables are dense



$$f = \bigvee_{c \,:\, \text{よ}_c \leq f} \text{よ}_c .$$

Thm (Universal property of the presheaf construction).

$$\begin{tikzcd} P \arrow[r, "f"] \arrow[d, "\text{よ}"'] & Q \\ \mathbb{2}^{P^{op}} \arrow[ur, "\hat{f}"'] & \end{tikzcd} \qquad Q \text{ has } \bigvee$$

$\forall f \ \forall Q$ (with $\bigvee$)

$\exists! \ \hat{f}$ preserve joins

$f = \hat{f} \circ \text{よ}$

Characterization

P has suprema iff

$$\begin{tikzcd} P \arrow[r, hook, "\text{よ}"'] & \mathbb{2}^{P^{op}} \arrow[l, dashed, bend right, "r"'] \end{tikzcd}$$

has a retraction r. $\qquad r \circ \text{よ} = 1.$

<u>Rem</u> $a, b \in \mathcal{C}$

$$(a, b) \longmapsto \mathcal{C}(a, b).$$

$$\longrightarrow \quad \begin{tikzcd} a \arrow[r, "s"] \arrow[dr, dashed] & b \arrow[d, "f"] \\ & b' \end{tikzcd} \longmapsto \begin{tikzcd} \mathcal{C}(a, b) \ni s \arrow[d, "f \circ -"] \\ \mathcal{C}(a, b') \quad f \circ s \end{tikzcd}$$

b fixed

$$\begin{tikzcd} a \arrow[d, "g"'] & \\ a' \arrow[r, "d"'] & b \end{tikzcd} \qquad \begin{tikzcd} \mathcal{C}(a, b) \quad d \circ g \\ \mathcal{C}(a', b) \arrow[u, "(-) \circ g"] \quad d \end{tikzcd}$$

$\forall c$ in $\mathcal{C}$

$\mathcal{C}(c, -) : \mathcal{C} \longrightarrow \mathrm{Set}$

$$\begin{tikzcd} d \arrow[r, mapsto] \arrow[d, "f"'] & \mathcal{C}(c, d) \arrow[d, "f \circ -"] \\ d' & \mathcal{C}(c, d') \end{tikzcd}$$

bump functions.

$\mathcal{C}(-, c) : \mathcal{C}^{op} \longrightarrow \mathrm{Set}.$

$$\begin{tikzcd} d \arrow[r, mapsto] \arrow[d, "g"'] & \mathcal{C}(d, c) \\ d' & \mathcal{C}(d', c) \arrow[u, "- \circ g"'] \end{tikzcd}$$

Example $\mathrm{id} : \mathrm{Set} \longrightarrow \mathrm{Set}$.

$$\mathrm{id} \cong \mathrm{Set}(1, -)$$

$$X \cong \mathrm{Set}(1, X).$$

$$x \longleftrightarrow 1 \mapsto x$$

$$\mathrm{Top} \xrightarrow{\mathcal{U}} \mathrm{Set}.$$

$$\mathcal{U} \cong \mathrm{Top}(1, -)$$

$$\mathcal{U}(X) = \mathrm{Top}(1, X).$$

$$\mathcal{U}(X) = |X|$$

$$\mathrm{Grp} \xrightarrow{\mathcal{U}} \mathrm{Set}$$

$$\mathcal{U} \cong \mathrm{Grp}(\mathbb{Z}, -)$$

$$\mathbb{Z} \cong F(1).$$

$$\mathcal{U}(G) \cong \mathrm{Grp}(\mathbb{Z}, G)$$

$$1 \mapsto g.$$

$$\mathcal{U}(G) = |G|$$

obj "forgetful functors or of this kind" "representable".

$$\mathcal{C}(c, -) \qquad \mathcal{C}(-, c).$$

$$\mathrm{Fld} \xrightarrow{\;\mathcal{U}\;} \mathrm{Set}.$$

is not representable.

does not have a left adjoint.

<u>Hint</u>

$$\mathcal{U}(X) \cong \mathrm{Set}(1, \mathcal{U}(X))$$

$\cong$ $F \dashv \mathcal{U}$ (is adjunction)

$$\mathcal{C}(F(1), X).$$

$$\mathcal{U}(-) \cong \mathcal{C}(F(1), -).$$

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pointed topological

homotopy classes of maps between topological spaces

\begin{tikzcd}
h\mathrm{Top}_\bullet \arrow[r, "\pi_1"] \arrow[rr, bend left, "\pi_1"] & \mathrm{Grp} \arrow[r, "\mathcal{U}"] & \mathrm{Set}
\end{tikzcd}

$$\pi_1 \cong h\mathrm{Top}_\bullet(S_1, X).$$

\begin{tikzcd}
\mathrm{Set}^{op} \arrow[r, "\mathcal{P}(-)"] & \mathrm{Set} \\
X \arrow[r, mapsto] \arrow[d, "f"'] & \mathcal{P}(X) \\
Y & \mathcal{P}(Y) \arrow[u, dashed, "f^{-1}"']
\end{tikzcd}

$$\mathcal{P}(-) \cong \underline{\mathrm{Set}(-, 2)}$$

<u>Thm</u> $\mathcal{P}(X) \cong 2^X.$

$$\mathrm{Top}^{op} \longrightarrow \mathrm{Frames} \longrightarrow \mathrm{Set}.$$

$$\begin{tikzcd}
X \arrow[r, mapsto] \arrow[d, "f"'] & \mathcal{O}(X) \\
Y & \mathcal{O}(Y) \arrow[u, "f^{-1}"']
\end{tikzcd}$$

$\mathcal{O}(X)$ ← open sets (topology of X)

$$\mathcal{O} \simeq \mathrm{Top}(-, S)$$

S ← Sierpinski

Obs

$$\langle -,- \rangle : P^{op} \times P \longrightarrow \mathbb{T}.$$

$$\mathcal{C}(-,-) \quad \mathcal{C}^{op} \times \mathcal{C} \longrightarrow \mathrm{Set}.$$

$$\begin{array}{ccc} (c, d) & \longmapsto & \mathcal{C}(c,d). \\ f \downarrow \quad \downarrow g & & \downarrow \; g \circ - \circ f. \\ c', d' & & \mathcal{C}(c', d') \end{array}$$

Discovery the construction is functorial!

The Yoneda embedding

$$\mathcal{C} \xrightarrow{\mathcal{Y}} \mathrm{Set}^{\mathcal{C}^{op}}$$

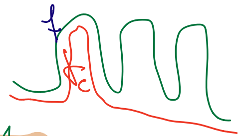
$$c \longmapsto \mathcal{C}(-, c).$$

$\mathcal{Y}$ are just "bump" functionals over $\mathcal{C}$.

obj : functors $\mathcal{C}^{o} \to \mathrm{Set}$

morphisms $F \overset{\alpha}{\Longrightarrow} G$.

Yoneda lemma



$$よ_c \leq f \iff f(c) = 1.$$

<u>thm</u> (Yoneda). there is a natural bijection between

$$\mathbf{Set}^{\mathcal{C}^{op}}(よ_c, X) \cong X(c).$$

$$\mathbf{Set}^{\mathcal{C}^{op}}(よ_c, X) \xrightarrow{\ \widehat{(-)}\ } X(c)$$

$$\underbrace{\alpha : よ_c \Rightarrow X}_{\text{evaluate in } d.} \longmapsto \alpha_c(1_c) =: \widehat{(\alpha)}$$

$$\alpha_d : よ_c(d) \longrightarrow X(d).$$

$$よ_c(d) = \mathcal{C}(d, c).$$

$$\alpha_c : よ_c(c) \longrightarrow X(c)$$

$$よ_c(c) = \mathcal{C}(c, c) \ni 1_c \qquad X(c) \ni \alpha_c(1_c)$$

$$
\begin{aligned}
X(c) &\longrightarrow \mathbf{Set}^{\mathcal{C}^{op}}(よ_c, X). \\
x &\longmapsto \bar{x} : よ_c \Longrightarrow X
\end{aligned}
$$

$$
\begin{aligned}
\bar{x}_b : よ_c(b) &\longrightarrow X(b). \\
\| \quad & \\
\mathcal{C}(b, c) &\longrightarrow X(b) \\
f &\longmapsto X(f)(x)
\end{aligned}
$$

(Sanity check)

$$
\begin{tikzcd}
X(c) \arrow[r] \arrow[rr, bend left, "1"] & \mathbf{Set}^{\mathcal{C}^{op}}(よ_c, X) \arrow[r] & X(c)
\end{tikzcd}
$$

$$
x \longmapsto \bar{x} \longmapsto \bar{x}_c(1_c) = 1_c(x) = x.
$$

thm $\mathcal{C} \xrightarrow{\text{よ}} \mathrm{Set}^{\mathcal{C}^{op}}$ is fully faithful.

full + faithful

$$\text{よ} : \mathcal{C}(a, b) \longrightarrow \mathrm{Set}^{\mathcal{C}^{op}}(\text{よ}a, \text{よ}b).$$

faithful — injective.

full — surjective.

$$\begin{tikzcd} d \arrow[d, "\exists g"'] & F(d) \arrow[d, "\forall s"'] \\ c & F(c) \end{tikzcd} \qquad s = F(g).$$

fullness.

Proof

$$\begin{tikzcd} \text{よ} : \mathcal{C}(a, b) \arrow[r, Rightarrow, "\text{よ}"] \arrow[dr, "\mathrm{id}"'] & \mathrm{Set}^{\mathcal{C}^{op}}(\text{よ}a, \text{よ}b) \\ & \mathcal{C}(a, b), \end{tikzcd}$$

$\mathrm{Set}^{\mathcal{C}^{op}}(\text{よ}a, \text{よ}b) \cong \text{よ}_b(a) \cong \mathcal{C}(a, b)$ — Yoneda lemma

Cor $a \cong b \iff f(a) \cong f(b)$.

Cor $\mathrm{Set} \xrightarrow{id} \mathrm{Set}$ has no non trivial natural transformation

$$\frac{\alpha : id \Longrightarrow id.}{\alpha : 1 \longrightarrow 1.} \text{ Yoneda}$$

so it must be the identity.

Yoneda ninja	Universal property	~~Charact of cocompleteness~~
		small presheaves
		$\boxed{\mathcal{P}(C)} \subset \mathrm{Set}^{C^{op}}$.