Previously

$A(Lx, y) \cong B(x, Ry)$

Adjunctions

Cantor

Yoneda

Basics

(Co) limits

$$\begin{tikzcd} P \arrow[r, dashed] \arrow[d, dashed] & A \arrow[d] \\ C \arrow[r] & B \end{tikzcd}$$

downgrade

- Monads
- Kan extensions
- Enriched category theory.

today

✓ Closure operators
✓ (posetal) Kan extensions
. Interaction

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① Monads (& closure operators).

Def A monad on a poset P is an endofunctor $T : P \to P$ such that

1) $p \leq Tp$ $\qquad 1 \xrightarrow{\eta} T$

2) $T_p \leq 1_p$. $\qquad T^2 \xrightarrow{\mu} T$

+ axiom

<u>Rmk</u> $T^2 = T$

1) $\Rightarrow T_p \leq T^2_p$.

2) $\Rightarrow T^2_p \leq T_p$

$\Rightarrow T^2 = T$.

<u>Rmk</u> The original name was triple (T, η, μ).

<u>Exam 1</u> X is a topological space

$cl : \mathcal{P}(X) \longrightarrow \mathcal{P}(X)$. $\qquad (\mathcal{P}(X), \subseteq)$.

functoriality

- $X \subset Y \Rightarrow cl(X) \subset cl(Y)$.
- $X \subset cl(X)$
- $cl^2 = cl$

monad.

(K) $cl(X \cup Y) = cl(X) \cup cl(Y)$.

the monad preserve finite joins.

<u>Example 2</u> Let V be a vector space.

$$\begin{aligned} \mathrm{Span} : \mathcal{P}(V) &\longrightarrow \mathcal{P}(V) \\ X &\longmapsto \mathrm{Span}(X). \end{aligned}$$

$$\longrightarrow \begin{cases} X \subset \mathrm{Span}(X) \\ \mathrm{Span}\,\mathrm{Span}(X) \subset \mathrm{Span}(X) \end{cases}$$

$$\begin{aligned} \langle - \rangle : \mathcal{P}(G) &\longrightarrow \mathcal{P}(G) \\ X &\longmapsto \langle X \rangle . \end{aligned}$$

G group.

$$X \longmapsto \begin{matrix} \mathrm{dcl}(X) \\ \mathrm{acl}(X). \end{matrix}$$

T first order theory

<u>Question</u>

<u>Motto</u> Algebraic Monads preserve directed unions.

$X_i \subset X_j \subset \cdots$

$$\mathrm{Span}\left(\bigcup X_i\right) = \bigcup \mathrm{Span}\, X_i$$

Monads

algebraic ← → topological nature

Def An algebra for a monad T on a poset P is an element p such that

$$Tp \leq p.$$

$$Tp \xrightarrow{a} p$$
+ axioms.

Rmk p is an algebra $\Leftrightarrow$ p is a fixed point.

Proof

$$\begin{array}{ccc} p & \leq & Tp \\ \| & & \wedge\!| \\ & & p \end{array}$$

(1) Axiom of monad

$$Tp = p.$$

Def Given a monad T on P we define the poset of algebras $Alg(T)$.

$$Alg(T) \xhookrightarrow{i} P$$

Prop The inclusion $Alg(T) \overset{i}{\hookrightarrow} P$ always has a left adjoint.

$$\text{"}T\text{"} : P \rightleftarrows Alg(T) : i \,.$$

Proof

$$\underline{Alg(T)(Tx, y)} \cong P(x, iy) \,.$$

$$cl\, x \leq y \iff x \leq iy$$

$\Longrightarrow$ obv. $\quad x \leq cl(x) \leq y$

$\Longleftarrow$ $\quad x \leq iy \Rightarrow cl(x) \leq \underbrace{cl(y)}_{= y}$

y is an algebra

$$cl(x) \leq y \,.$$

Rem $i : Alg(T) \longrightarrow P$ preserve all limits. (it's a right adjoint).

X topological space

$Alg(cl) =$ [illegible]

$Alg(cl) =$ fixed points of cl
$\|$
closed sets.

G is group

$Alg(\langle\, \rangle) =$ subgroups.

V is a vect space

$Alg(\mathrm{Span}) =$ subspaces.

Rmk " $Cl(X) \hookrightarrow \mathcal{P}(X)$
closed sets are "closed" under
arbitrary intersection.
$\|$
is an instance of $i : Alg(T) \to P$
preserve all limits.

$T : P \to P$

$p \leq Tp$

$T^2 p \leq p$

We care about algebras

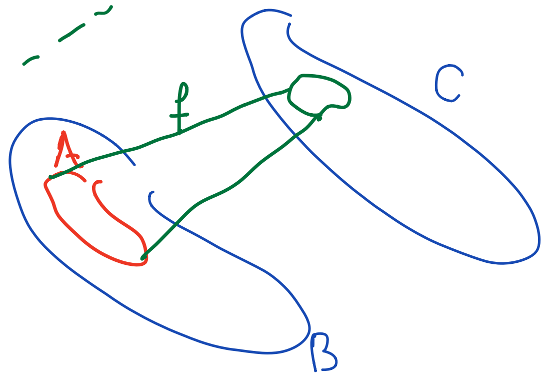
$Alg(T) \hookrightarrow P$.

$P' \hookrightarrow P$.

Kan extensions

inclusion

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, hook, "i"'] & C \\
B \arrow[ur, dashed, "\hat{f}"'] &
\end{tikzcd}$$



$\hat{f}$ extending f.

Naive student of the 1st year?

$$\hat{f}(x) = \boxed{f \circ i^{-1}(x)}$$

- $i^{-1}(x)$ might be empty.
- $i^{-1}(x)$ might have many elements.

C is a complete poset.

$$\hat{f}(x) \quad \begin{cases} \rightsquigarrow \sup\limits_{y \leq i^{-1}(x)} f(y). \\ \rightsquigarrow \inf\limits_{y \geq i^{-1}(x)} f(x). \end{cases}$$

What did we do?

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "i"] & C \\
B \arrow[ur, dashed, "\mathrm{lan}_i f"] \arrow[d, "i^{-1}"] & \\
\mathcal{P}(A) \arrow[r, "f_!"] & \mathcal{P}(C) \arrow[uu, "\sup"']
\end{tikzcd}$$

left Kan extension

Rem

$$\mathrm{lan}_i f = \sup \circ \mathcal{P}_!(f) \circ i^{-1}$$

$$\rightsquigarrow \mathrm{ran}_i f = \inf \sim\sim\sim\sim$$

Rem concrete computation;

$$\begin{tikzcd} \mathrm{Alg}(T) \arrow[r, equal] \arrow[d, hook, "i"'] & \mathrm{Alg}(T) \\ p \in P \arrow[ur, dashed, "\mathrm{ran}_i(1)"'] \arrow[ur, phantom, "T", near start, shift left=6] & \end{tikzcd}$$

$$\mathrm{ran}_i(1)(p) = \inf_{p \leq i(x)} x$$

$$= \inf_{Tp \leq x} x$$

$$= Tp.$$

$$\mathrm{ran}_i(1) = T,$$

Rem $\mathrm{ran}_i\, i$ is a monad!

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "i"'] & B \\ B \arrow[ur, dashed, "\mathrm{ran}_i\, i"'] & \end{tikzcd}$$

$$\begin{tikzcd} \mathrm{Pos}/P \arrow[r, "\mathrm{ran}_{(-)}(-)"] & \mathrm{Monads}(P) \end{tikzcd}$$

$$\mathrm{Pos}/P = \left\{ P' \longrightarrow P \right\}$$

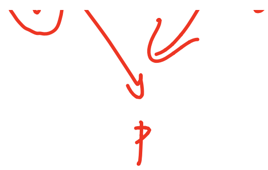
P' is a poset

$$\begin{tikzcd} \mathrm{Pos}/P & \mathrm{Monad}(P) \arrow[l, bend left, "\mathrm{Alg}(-)"] \end{tikzcd}$$

$$\mathrm{Alg}(T) \hookrightarrow P$$

$$\begin{tikzcd} P' \arrow[d, hook, "i"] & \rightsquigarrow \ \mathrm{ran}_i\, i \ \rightsquigarrow & \mathrm{Alg}(\mathrm{ran}_i\, i) \arrow[d, hook] \\ P & & P \end{tikzcd}$$

$$\begin{tikzcd} P' \arrow[r, "\tilde{\theta}"] \arrow[d, "i"'] & \mathrm{Alg}(\mathrm{ran}_i\, i) \arrow[dl, "j"] \\ {} & \end{tikzcd}$$



Thm (Lazy Barr-Beck monadicity)

$C \xhookrightarrow{i} P$, P complete

is monadic iff

- C is complete
- i preserve inf.
- i is injective.