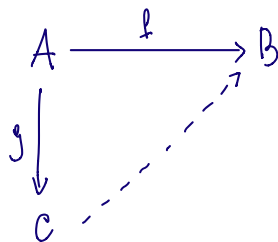


# Kan extensions

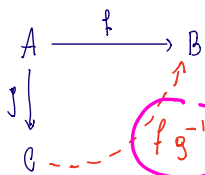
The extension problem



Many universal construction can be encoded as "solution to an extension problem", -

We will use this theory to construct adjoints (Adjoint functor theorem).

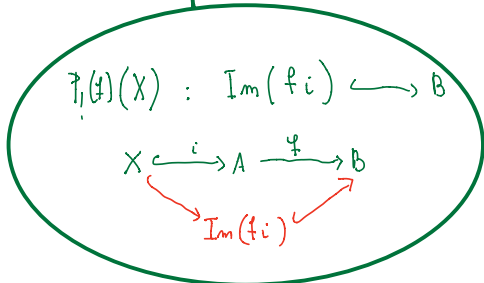
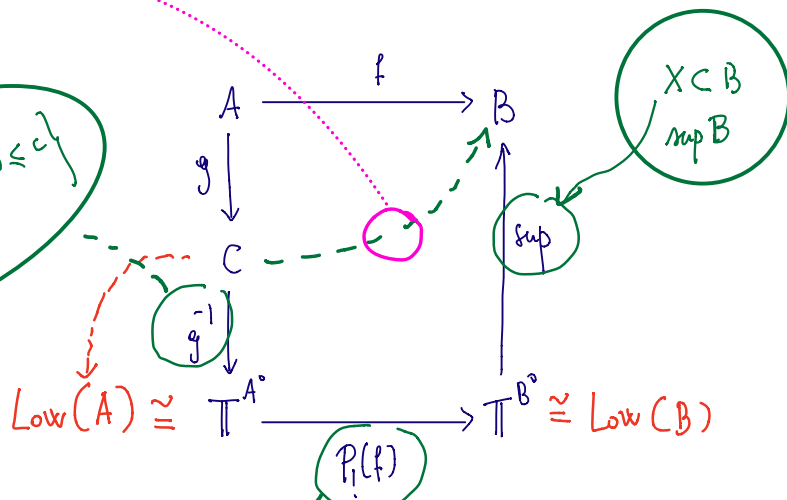
## Intuition



What would you do for posets?

(If B has suprema)

$$g^{-1}(c) = \{a \in A : g(a) \leq c\}$$



So, if we want to follow this program, we need to give a definition of several ingredients for cats -

- $\mathcal{P}(A)$ , a substitute for  $\mathbb{T}^{\text{po}}$
- $g^{-1}$
- $\mathcal{P}_!(\mathbb{I})$
- $\text{sup?}$

①  $\mathcal{P}(A)$ , (the category of "small" presheaves)

Def Given a locally small category  $A$  we define  $\mathcal{P}(A)$  the full subcategory of  $\text{Set}^{A^{\text{op}}}$  of those presheaves that are small colimits of representables

We will see where this is relevant

$$\text{Yonk } \mathbb{I}_A : A \longrightarrow \mathcal{P}(A) \quad (\text{Yoneda})$$

1.1) the Grothendieck construction

In the analogy that we are pushing,  
 every  $f: \mathbb{P}^{\mathbb{P}} \rightarrow \mathbb{P}$  induces a lower  
 set " $f^{\downarrow} \hookrightarrow \mathbb{P}$ ".

$$\mathbb{P}^{\mathbb{P}} \rightsquigarrow \text{Low } \mathbb{P}$$

Also the small presheaf construction  
 has this property (some how) -

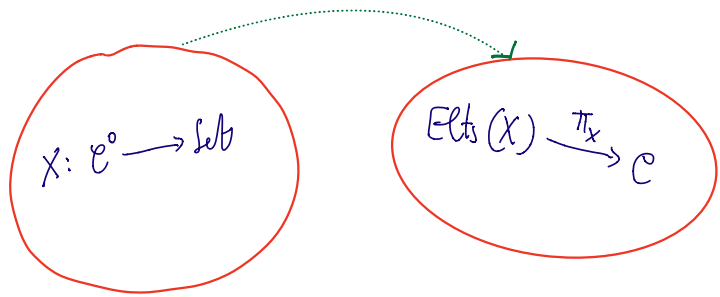
Def For a presheaf  $X: \mathcal{C}^{\circ} \rightarrow \text{Set}$   
 we define its **category of elements**

$\text{Elt}_s(X) \rightarrow \text{obj } (c, a_c): c \in \mathcal{C}, a_c \in X(c).$

$\rightarrow \text{arrow } f: c \rightarrow d \in \mathcal{C}(c, d) \text{ such}$   
 that  $X(f)(a_c) = a_d$ .

Rem there is a natural projection

$$\text{Elt}_s(X) \xrightarrow{\pi_X} \mathcal{C}$$



Rem  $\pi_X$  is not a full functor!

Rem We could characterize the image, but we do not  $\Rightarrow$  -  
(Discrete opfibration) -

Rem One can also go in the other direction

$$\begin{array}{c} \pi: D \xrightarrow{X} \mathcal{C} \\ \Downarrow \\ \pi: D \xrightarrow{X} \mathcal{C} \xrightarrow{f} \text{Set}^{\mathcal{C}^{\circ}} \\ \hline \pi_f := \text{colim}(f \circ X) \in \text{Set}^{\mathcal{C}^{\circ}} \end{array}$$

$$\begin{array}{ccc} & \text{colim}(f \circ -) & \\ & \swarrow & \\ \text{Set}^{\mathcal{C}^{\circ}} & & \text{Cat}/\mathcal{C} \\ & \searrow & \\ & \text{Elt}_s(-) & \end{array}$$

Thm  $\text{colim}(f \circ \pi_p) \cong \Phi$

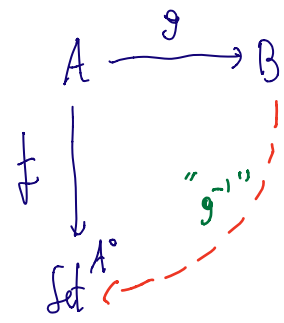
② "  $-1$  "

$\mathcal{P}(A)$ , a substitute for  $\mathbb{T}^{op}$

- $g^*$
- $\mathcal{P}(g)$
- sup?

Def the "kernel" of  $g$  is the functor

$$B(g-, -): B \longrightarrow \text{Set}^{A^{op}}$$

$$b \longmapsto B(g-, b)$$


Rule this is well defined, indeed

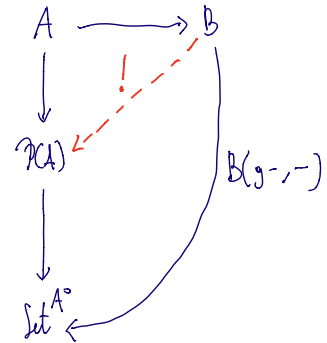
$$B(g-, b): A^{op} \longrightarrow \text{Set}$$

$$a \longmapsto B(ga, b)$$

$$A \xrightarrow{g} B$$

$$a \longmapsto ga$$

Def A functor is **admissible** if for every  $b \in B$ ,  $B(g-, b)$  is a small presheaf.

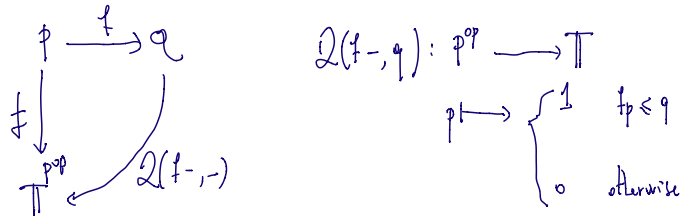


Rule functors with domain small categories are admissible

Rule functors with arity are admissible.

Rule accessible functors are admissible

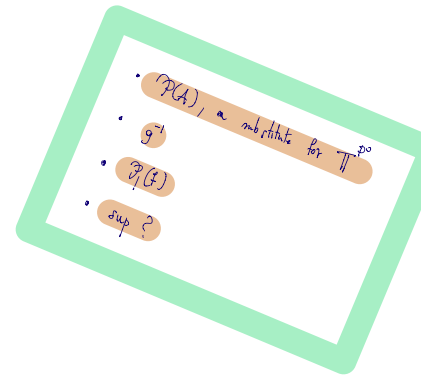
Run ( sanity check, the inverse  
in posets is a weak  
"counterimage" -



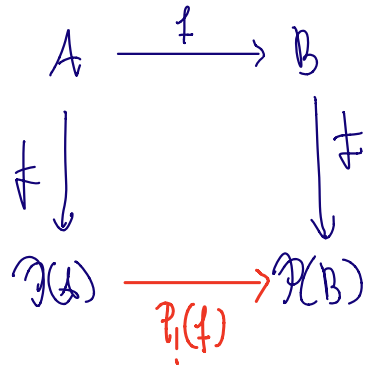
So the corresponding lower set  
to  $\mathcal{Q}(f, q)$  is

$$"f^{-1}(q)" = \{ p \in P : fp \leq q \}$$

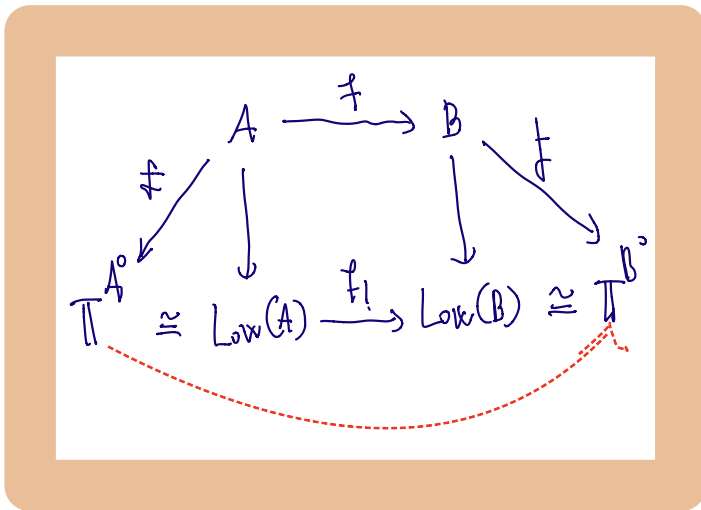
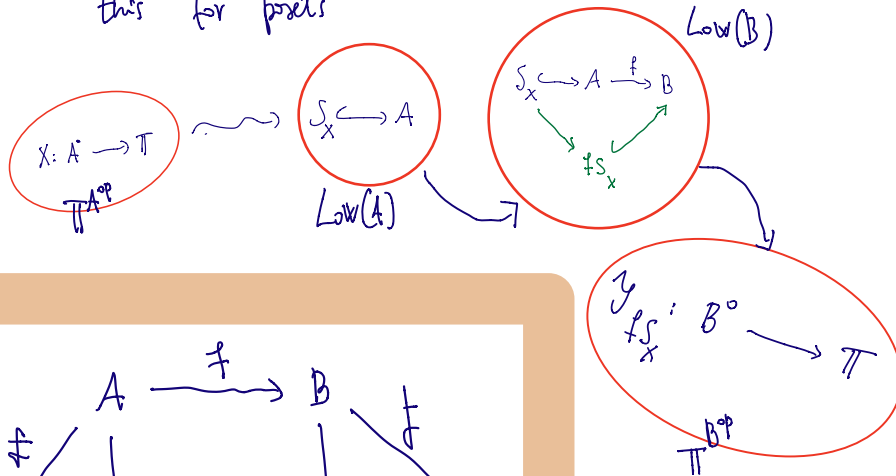
③  $\mathcal{P}_!$



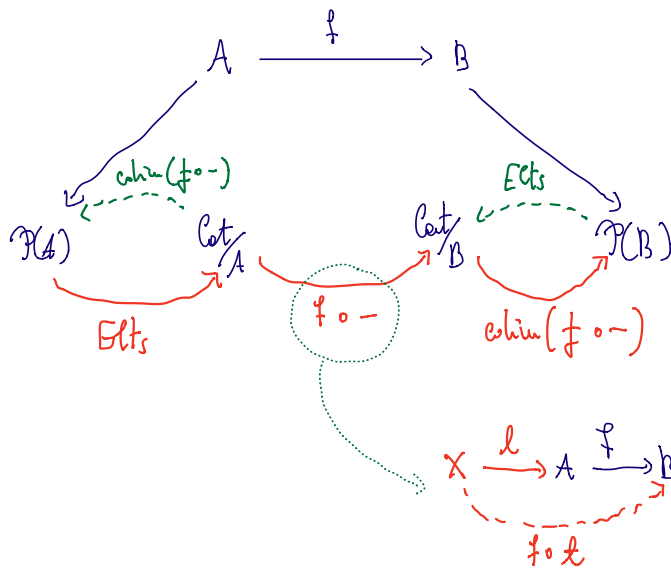
So, now, given a functor  
 $f: A \longrightarrow B$  we would like  
to construct  $\mathcal{P}_!(f): \mathcal{P}(A) \longrightarrow \mathcal{P}(B)$ .



Now of course we know how to do this for posets



Now we just do the same!



Run the idea is

$$P_f(\pm)(X) = P_f(f)\left(\bigcup_{x \in X} \{x\}\right) = \bigcup_{x \in X} \{f(x)\}.$$

(3) sup

$P(A)$ , a substitute for  $\mathbb{T}^{\text{po}}$   
 $g'$   
 $P_f(f)$   
 sup?

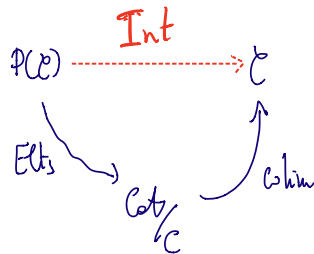
Thm If  $\mathcal{C}$  is a small-cocomplete category, the Yoneda embedding has a left adjoint

$$\text{Int} : \mathcal{P}(\mathcal{C}) \rightleftarrows \mathcal{C} : f$$

Proof

we build Int.

Run this is where we use the presheaves are small, to cut down the size of the colimit



Now we prove that Int is left adjoint. I want to show that...

$$\mathcal{C}(\text{Int} X, d) \cong \mathcal{P}(\mathcal{C})(X, fd)$$

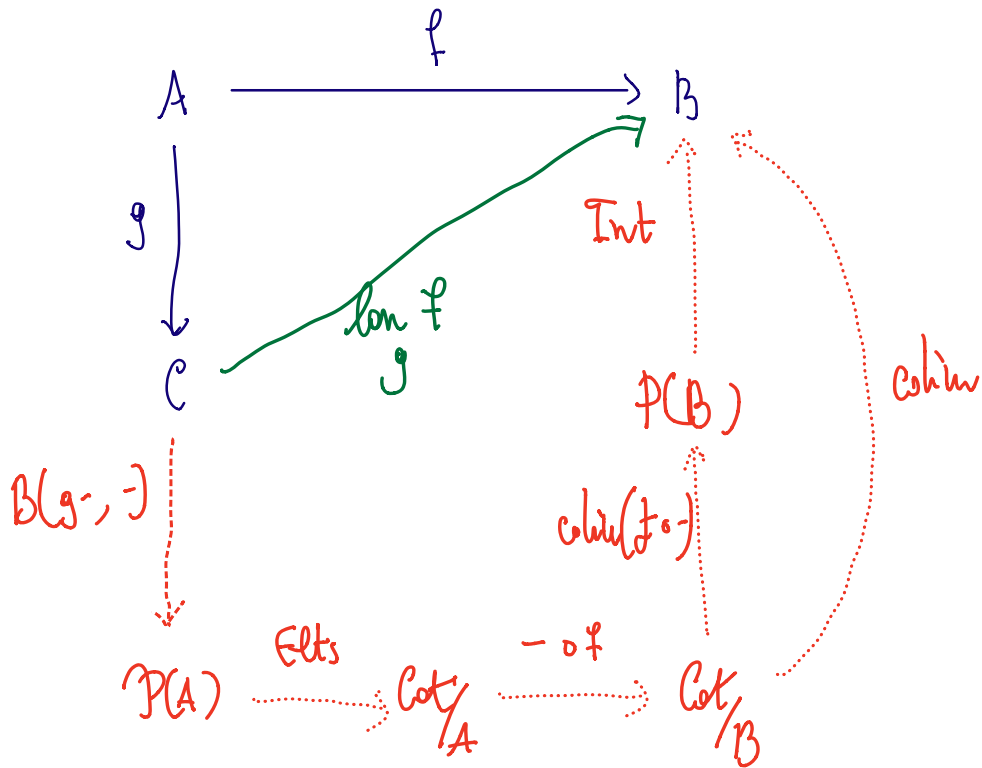


Now,  $\mathcal{P}(C)(x, f \downarrow) \cong \mathcal{P}(C)(\text{clim}(f \circ \pi_x), f \downarrow)$

$\mathcal{P}(\text{Int} X, d) \stackrel{\text{J.L.}}{\cong} \lim \mathcal{P}(C)(f \circ \pi_x, f \downarrow)$

Wrapping up!

Thm If  $f$  is an admissible function on  $B$  is complete, we can extend!



Now let us see some properties of this construction.

① We have constructed **left** Kan extension, using limits one can construct **right** Kan extensions

② There is a more general presentation, which is useful in more abstract category theory. We are **very** **concrete**.

Defn Consider a functor  $g: A \rightarrow C$   
then this induces a functor

$$[A, B] \xleftarrow{g^*} [C, B]$$

$$f \circ g \longleftarrow f$$

then " $\text{lan}_g$ ":  $[A, B] \longrightarrow [C, B]$   
is left adjoint to  $g^*$ .

$$\text{lan}_g: [A, B] \xrightleftharpoons{\quad} [C, B] : g^*$$

Ex try to find the unit  $\eta$  and counit!

Prop if  $g$  is fully faithful,  
the Kan extension is an extension, i.e.

$$\left( \underset{g}{\text{lan } f} \right) \circ g \cong f$$

Application:  $A \nrightarrow B$

Lemma (would be easy with the  
more axiomatic presentation of  
ex) Let  $f: A \rightarrow B$  be  
a cocontinuous functor between  
cocomplete categories. Then  $\text{RAN } f$ .

①  $f$  has a right adj.

②  $\text{lan } 1$  exists and is the right adj.

Cor  
Adj  
functor  
Theorem

Let  $f: A \longrightarrow B$  be  
a functor between complete  
categories. **TFAE.**

- ①  $f$  is **admissible and cocontinuous**
- ②  $f$  **has a right adjoint**

Proof

1  $\Rightarrow$  2) ok, this is a corollary  
of the previous part of the  
lesson.

2  $\Rightarrow$  1) left adjoints are cocontinuous.  
we need to show that it is  
admissible. that is  $f \dashv g$

$\forall b \in B$ ,  $B(f-, b)$  is  
a small colimit of representables.

Now  $B(f-, b) \cong A(-, gb)$

But then

$$B(f-, b) \cong \underline{f}(gb)$$

so it is representable!

# CATEGORY THEORY

IVAN DI LIBERTI

## EXERCISES

**Riehl** (Kan extensions have a universal property). Read section 6.1, where a Kan extensions are introduced in a more abstract way and study Thm 6.2.1 which proves that our concrete formula is explicitly computing the Kan extension, when possible.

**Riehl** (Concepts are Kan extensions). Read section 6.5, where it is shown that many categorical concepts can be phrased in terms of existence of Kan extensions.

**Exercise 1** (▣). Prove<sup>a</sup>, when all the functors in the equations are well-defined, that

$$\text{lan}_{fg}(h) \cong \text{lan}_f(\text{lan}_g h).$$

**Exercise 2** (▣). Try to show that if  $f$  has a right adjoint  $g$ , then

$$\text{lan}_f(1) \cong g.$$

**Exercise 3** (▣). Prove, using our definition, that when  $g$  is fully faithful, then  $(\text{lan}_g f) \circ g \cong f$ .

<sup>a</sup>Hint. Use that Kan extensions provide left adjoints to precomposition.

- the exercises in the red group are mandatory.
  - pick at least one exercise from each of the yellow groups.
  - pick at least two exercises from each of the blue groups.
  - nothing is mandatory in the brown box.
  - The riddle of the week. It's just there to let you think about it. It is not a mandatory exercise, nor it counts for your evaluation. Yet, it has a lot to teach.
  - ☞ useful to deepen your understanding. Take your time to solve it. (May not be challenging at all.)
  - ▣ measures the difficulty of the exercise. Note that a technically easy exercise is still very important for the foundations of your knowledge.
  - ⚠ It's just too hard.
- The label **Leinster** refers to the book **Basic Category Theory**, by *Leinster*.  
The label **Riehl** refers to the book **Category Theory in context**, by *Riehl*.