Kan extensions

The extension problem

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \\ C \arrow[ur, dashed] & \end{tikzcd}$$

Many universal construction can be encoded as "solution to an extension problem".

We will use this theory to construct adjoints (Adjoint functor theorem).

Intuition

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \\ C \arrow[ur, dashed, "fg^{-1}"'] & \end{tikzcd}$$

What would you do for Posets? (If B has supreme)

$$\begin{tikzcd} & A \arrow[r, "f"] \arrow[d, "g"'] & B & \\ & C \arrow[ur, dashed] \arrow[d, "g^{-1}"'] & & \\ \mathrm{Low}(A) \cong & \mathbb{T}^{A^\circ} \arrow[r, "P_!(f)"'] & \mathbb{T}^{B^\circ} \arrow[uu, "\sup"'] & \cong \mathrm{Low}(B) \end{tikzcd}$$

$g^{-1}(c) = \{a \in A : g(a) \leq c\}$

$X \subset B$
$\sup B$

$P_!(f)(X) : \mathrm{Im}(fi) \hookrightarrow B$

$$\begin{tikzcd} X \arrow[r, hook, "i"] \arrow[dr] & A \arrow[r, "f"] & B \\ & \mathrm{Im}(fi) \arrow[ur] & \end{tikzcd}$$

So, if we want to follow this program, we need to give a definition of several ingredients for cats.

- $\mathcal{P}(A)$, a substitute for $\Pi^{p^{o}}$
- g^{-1}
- $\mathcal{P}_!(f)$
- sup ?

① $\underline{\mathcal{P}(A)}$, (the category of "small" presheaves)

Def Given a locally small category A we define $\mathcal{P}(A)$ the full subcategory of $\mathrm{Set}^{A^{op}}$ of those presheaves that are small colimits of representables.

We will see where this is relevant

Rmk $\text{よ}_A : A \longrightarrow \mathcal{P}(A)$ (Yoneda)

1.1 The Grothendieck construction

In the analogy that we are pushing, every $f : P^{op} \longrightarrow \mathbb{T}$ induces a lower set "$f^{-1}(1) \hookrightarrow P$".

$$\mathbb{T}^{P^o} \rightsquigarrow \mathrm{Low}\,P$$

Also the small presheaf construction has this property (somehow) -

Def For a presheaf $X : \mathcal{C}^o \longrightarrow \mathrm{Set}$ we define its **category of elements**

$\mathrm{Elts}(X)$ $\rightsquigarrow$ obj (c, a_c): $c \in \mathcal{C}$, $a_c \in X(c)$.

$\rightsquigarrow$ arrow $f : c \to d \in \mathcal{C}(c,d)$ such that $X(f)(a_c) = (a_d)$.

<u>Rem</u> There is a natural projection
$$\mathrm{Elts}(X) \xrightarrow{\pi_X} \mathcal{C}$$

$$X : \mathcal{C}^o \longrightarrow \mathrm{Set} \quad \rightsquigarrow \quad \mathrm{Elts}(X) \xrightarrow{\pi_X} \mathcal{C}$$

Rem π_X is not a full functor!

Rem We could characterize the image, but we do not =).
(Discrete opfibration).

Rem One can also go in the other direction

$$\frac{\dfrac{\pi : D \xrightarrow{X} \mathcal{C}}{\pi : D \xrightarrow{X} \mathcal{C} \xrightarrow{よ} \mathrm{Set}^{\mathcal{C}^{o}}}}{P_\pi := \mathrm{colim}(よX) \in \mathrm{Set}^{\mathcal{C}^{op}}.}$$

$$\begin{tikzcd}
\mathrm{Set}^{\mathcal{C}^{op}} \arrow[r, bend right, "\mathrm{Elts}(-)"'] & \mathrm{Cat}/\mathcal{C} \arrow[l, bend right, "\mathrm{colim}(よ\circ -)"']
\end{tikzcd}$$

$$\boxed{\underline{\text{Thm}} \quad \mathrm{colim}(よ \circ \pi_P) \cong P}$$

② "g^{-1}".

- $\mathcal{P}(A)$, a substitute for $\prod^{po}$
- g^{-1}
- $\mathcal{P}_!(f)$
- sup ?

Def The "nerve" of g is the functor

$$B(g-,-) : B \longrightarrow Set^{A^{op}}$$
$$b \longmapsto B(g-,b)$$

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, "f"'] & B \arrow[dl, dashed, bend left, "\text{"}g^{-1}\text{"}"] \\ Set^{A^{o}} & \end{tikzcd}$$

Rmk this is well defined, indeed

$$B(g-,b) : A^{op} \longrightarrow Set$$
$$a \longmapsto B(ga, b)$$

$$A \xrightarrow{g} B$$
$$a \longmapsto ga.$$

Def A functor is admissible if for every $b \in B$, $B(g-,b)$ is a small presheaf.

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[dl, dashed, "!"'] \arrow[ddl, bend left, "B(g-,-)"] \\ \mathcal{P}(A) \arrow[d] & \\ Set^{A^{o}} & \end{tikzcd}$$

Rmk functors with domain small categories are admissible

Rmk functors with arity are admissible.

Rmk accessible functors are admissible

<u>Rmk</u> (sanity check, the nerve in posets is a weak counterimage" -

$$\begin{tikzcd} P \arrow[r, "f"] \arrow[d, "よ"'] & Q \arrow[dl, bend left, "Q(f-,-)"] \\ \Pi^{P^{op}} & \end{tikzcd}$$

$$Q(f-,q) : P^{op} \longrightarrow \Pi$$

$$p \longmapsto \begin{cases} 1 & fp \leq q \\ 0 & \text{otherwise} \end{cases}$$

So the corresponding lower set to $Q(f-,q)$ is

$$"f^{-1}(q)" = \{ p \in P : fp \leq q \}$$

3 $P_!$

- $\mathcal{P}(A)$, a substitute for $\Pi^{P^{o}}$
- g^{-1}
- $\mathcal{P}_!(f)$
- sup ?

So, now, given a functor $f : A \longrightarrow B$ we would like to construct $P_!(f) : \mathcal{P}(A) \longrightarrow \mathcal{P}(B)$.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "よ"'] & B \arrow[d, "よ"] \\ \mathcal{P}(A) \arrow[r, "P_!(f)"'] & \mathcal{P}(B) \end{tikzcd}$$

Now of course we know how to do this for posets

$$X : A^\circ \longrightarrow \mathbb{T} \quad \big(\mathbb{T}^{A^{op}}\big) \quad \rightsquigarrow \quad S_X \hookrightarrow A \quad \big(Low(A)\big) \quad \longrightarrow \quad \begin{tikzcd} S_X \arrow[r, hook] \arrow[dr] & A \arrow[r, "f"] & B \\ & f S_X \arrow[ur] & \end{tikzcd} \quad \big(Low(B)\big)$$

$$\longrightarrow \quad y_{fS_X} : B^\circ \longrightarrow \mathbb{T} \quad \big(\mathbb{T}^{B^{op}}\big)$$

$$\boxed{\begin{tikzcd} & A \arrow[r, "f"] \arrow[dl, "y"'] \arrow[d] & B \arrow[d] \arrow[dr, "y"] & \\ \mathbb{T}^{A^\circ} \arrow[r, phantom, "\cong"] \arrow[rrr, dashed, bend right] & Low(A) \arrow[r, "f_!"] & Low(B) \arrow[r, phantom, "\cong"] & \mathbb{T}^{B^\circ} \end{tikzcd}}$$

Now we just do the same!

$$\begin{tikzcd} & A \arrow[rr, "f"] \arrow[dl, "y"'] & & B \arrow[dr, "y"] & \\ \mathcal{P}(A) \arrow[r, bend right, "Elts"'] & Cat/A \arrow[l, dashed, bend right, "colim(y\circ -)"'] \arrow[rr, bend right, "f\circ -"'] & & Cat/B \arrow[r, bend right, "colim(y\circ -)"'] & \mathcal{P}(B) \arrow[l, dashed, bend right, "Elts"'] \end{tikzcd}$$

$$\begin{tikzcd} X \arrow[r, "\ell"] \arrow[rr, dashed, bend right, "f\circ\ell"'] & A \arrow[r, "f"] & B \end{tikzcd}$$

Rem the idea is

$$\mathcal{P}_!(f)(X) = \mathcal{P}_!(f)\Big(\bigcup_{x\in X}\{x\}\Big) = \bigcup_{x\in X}\{f(x)\}.$$

③ sup

- $\mathcal{P}(A)$, a substitute for Π^{op}
- g^{-1}
- $\mathcal{P}_!(f)$
- sup ?

Thm If $\mathcal{C}$ is a small-cocomplete category, the Yoneda embedding has a left adjoint

$$\mathrm{Int} : \mathcal{P}(\mathcal{C}) \rightleftarrows \mathcal{C} : よ$$

Proof We build Int.

$$\begin{tikzcd}
\mathcal{P}(\mathcal{C}) \arrow[rr, dashed, "\mathrm{Int}"] \arrow[dr, bend right, "\mathrm{Elts}"'] & & \mathcal{C} \\
& \mathrm{Cat}/\mathcal{C} \arrow[ur, bend right, "\mathrm{colim}"'] &
\end{tikzcd}$$

Rem this is where we use the presheaves are small, to cut down the size of the colimit

Now we prove that Int is left adjoint. I want to show that...

$$\mathcal{C}(\mathrm{Int}\,X, d) \cong \mathcal{P}(\mathcal{C})(X, よd)$$

Now, $\mathcal{P}(\mathcal{C})(X, f\downarrow) \cong \mathcal{P}(\mathcal{C})\big(\operatorname{colim}(f\circ\pi_X), f\downarrow\big)$

$$\mathcal{C}(\operatorname{Int} X, d) \overset{Y.L}{\cong} \lim \mathcal{P}(\mathcal{C})(f\circ\pi_X, f\downarrow)$$

Wrapping up!

Thm If f is an admissible functor or B is cocomplete, we can extend!

$$\begin{tikzcd}
A \arrow[rr, "f"] \arrow[d, "g"'] & & B \\
C \arrow[urr, "\operatorname{Lan}_g f"] \arrow[d, dashed, "B(g-,-)"'] & & \mathcal{P}(B) \arrow[u, dashed, "\operatorname{Int}"'] \\
\mathcal{P}(A) \arrow[r, dashed, "\operatorname{Elts}"] & \operatorname{Cat}/A \arrow[r, dashed, "-\circ f"] & \operatorname{Cat}/B \arrow[u, dashed, "\operatorname{colim}(f\circ -)"] \arrow[uu, dashed, bend right, "\operatorname{colim}"']
\end{tikzcd}$$

Now let us see some properties of this construction.

1. We have constructed left Kan extension, using limits. One can construct rigth Kan extensions

2. There is a more general presentation, which is useful in more abstract category theory. We are very concrete.

<u>Rem</u> Consider a functor $g : A \to C$ then this induces a functor

$$\begin{array}{ccc} [A, B] & \xleftarrow{\quad g^* \quad} & [C, B] \\ f \circ g & \longleftarrow\!\!\shortmid & f \end{array}$$

<u>Thm</u> "lan_g": $[A, B] \longrightarrow [C, B]$ is left adjoint to g^*.

$$\mathrm{lan}_g : [A, B] \rightleftarrows [C, B] : g^*$$

Ex try to find the unit & counit!

Prop if g is fully faithful, the Kan extension is an extension, i.e.

$$\left(\mathrm{Lan}_g f\right) \circ g \cong f$$

Application: AFT

Lemma (would be easy with the more axiomatic presentation of ex) Let $f : A \longrightarrow B$ be a cocontinuous functor between cocomplete categories. then TFAE.

1. f has a right adj
2. $\mathrm{Lan}_f 1$ exists and is the right adj.

Cor

Adj Functor Theorem

Let $f : A \longrightarrow B$ be a functor between cocomplete categories. TFAE.

1. f is admissible and cocontinuous
2. f has a right adjoint

Proof

$1 \Rightarrow 2)$ ok, this is a corollary of the previous part of the lesson.

$2 \Rightarrow 1)$ left adjoints are cocontinous, we need to show that it is admissible. that is

$f \dashv g$

$\forall\ b \in B$, $B(f-, b)$ is a small colimit of representables.

Now $B(f-, b) \cong A(-, gb)$

But then

$$B(f-, b) \cong よ(gb)$$

so it is representable!

CATEGORY THEORY

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EXERCISES

Riehl (Kan extensions have a universal property)**.** Read section 6.1, where a Kan extensions are introduced in a more abstract way and study Thm 6.2.1 which proves that our concrete formula is explicitely computing the Kan extension, when possible.

Riehl (Concepts are Kan extensions)**.** Read section 6.5, where it is shown that many categorical concepts can be phrased in terms of existence of Kan extensions.

Exercise 1 (🔋)**.** Prove[a], when all the functors in the equations are well-defined, that

$$\mathrm{lan}_{fg}(h) \cong \mathrm{lan}_f(\mathrm{lan}_g h).$$

Exercise 2 (🔋)**.** Try to show that if f has a right adjoint g, then

$$\mathrm{lan}_f(1) \cong g.$$

Exercise 3 (🔋)**.** Prove, using our definition, that when g is fully faithful, then $(\mathrm{lan}_g f)\circ g \cong f$.

[a] Hint. Use that Kan extensions provide left adjoints to precomposition.

- 🔴 the exercises in the red group are mandatory.
- 🟡 pick at least one exercise from each of the yellow groups.
- 🔵 pick at least two exercises from each of the blue groups.
- 🟤 nothing is mandatory in the brown box.
- 🟢 The riddle of the week. It's just there to let you think about it. It is not a mandatory exercise, nor it counts for your evaluation. Yet, it has a lot to teach.
- ☕ useful to deepen your understanding. Take your time to solve it. (May not be challenging at all.)
- 🔋 measures the difficulty of the exercise. Note that a technically easy exercise is still very important for the foundations of your knowledge.
- ⚠ It's just too hard.

The label **Leinster** refers to the book **Basic Category Theory**, by *Leinster*.
The label **Riehl** refers to the book **Category Theory in context**, by *Riehl*.

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