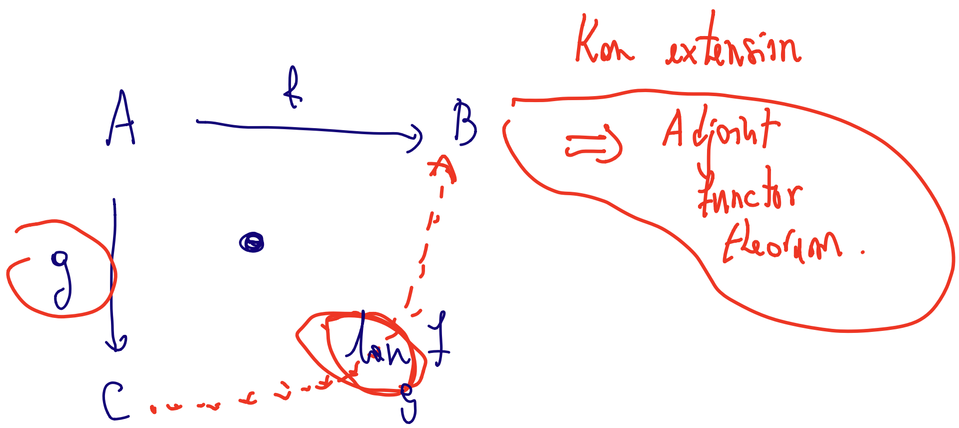
Kan extensions



"Concrete"
(Additional assumptions)
(Concrete formula)

Historically were faithful

"Axiomatic"
(is more general)
(there is no concrete formula to compute the extension)

a very conceptual understanding of the concrete construction.

Intuition

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "g"'] & B \\
C \arrow[ur, "f \circ g^{-1}"'] &
\end{tikzcd}$$

Posets.

$$\begin{tikzcd}
& A \arrow[r, "f"] \arrow[d, "g"'] & B \\
& C \arrow[ur, "\sup \circ P_!(f) \circ g^{-1}"'] \arrow[d, "g^{-1}"'] & \\
\mathrm{Low}(A) \cong & \Pi^{A^\circ} \arrow[r, "P_!(f)"] & \Pi^{B^\circ} \cong \mathrm{Low}(B) \arrow[u, "\sup"']
\end{tikzcd}$$

has supremum,

$\Pi = \{0 < 1\}$

"powerset".

$$P_!(f)(X) = P_!(f)\Big(\bigcup_{x \in X} \{x\}\Big)$$

$$\bigcup_{x \in X} f(x).$$

List of stuff to do in CT.

1. An analog of Π^{A° $(\mathcal{P}(A))$.
2. g^{-1} (an analog of).
3. $P_!(f)$
4. sup (?)

(1) $\mathcal{P}(A)$ some notion of powerset.

(the category of small presheaves).

Def Given a locally small category A, $\mathcal{P}(A)$ (small presheaves) is the full subcategory of $\mathrm{Set}^{A^{\circ}}$ containing small colimits of representable functors.

$$\begin{tikzcd} & A \arrow[dl, dashed, "y"'] \arrow[d, "y"] \\ \mathcal{P}(A) \arrow[r, hook, "i"] & \mathrm{Set}^{A^{\circ}} \end{tikzcd}$$

Rem if A is small i is equivalence!

Is this a good notion of powerset?

"The Grothendieck construction"

$$\mathrm{Low}(P) \simeq \Pi^{P^o}$$

Given a small presheaf $X : \mathcal{C}^o \longrightarrow \mathrm{Set}$

I want to construct a functor

$$\pi_X : \mathrm{Elts}(X) \longrightarrow \mathcal{C}$$

<u>Def</u> Given a presheaf $X : \mathcal{C}^o \longrightarrow \mathrm{Set}$ we define its category of elements

$\mathrm{Elts}(X)$

ob $= (c, a_c)$ where c is an object of $\mathcal{C}$ and a_c is an element of $a_c \in X(c)$.

$$X : \mathcal{C}^o \longrightarrow \mathrm{Set}$$
$$c \longmapsto X(c)$$

arr $=$ $(c, a_c) \longrightarrow (d, a_d)$ is a morphism in $\mathcal{C}$ between $c \xrightarrow{f} d$ such that

$$X(f)(a_d) = a_c.$$

<u>Rem</u> $\pi_X : \mathrm{Elts}(X) \longrightarrow \mathcal{C}$

$$(c, a_c) \longmapsto c$$
$$f \longmapsto f_-$$

$X : \mathcal{C}^{\circ} \longrightarrow \mathrm{Set}$ $\xrightarrow{\mathrm{Elts}}$ $\mathrm{Elts}(X) \xrightarrow{\pi_X} \mathcal{C}$ (Discrete opfibrations)

boolean functionals

subsets

$\mathcal{G}$

$\mathrm{colim}(\text{よ} \circ -)$

construct a presheaf. $\leftsquigarrow$ $F : A \longrightarrow \mathcal{C}$ $\in \mathrm{Cat}/_{\mathcal{C}}$

$A \overset{i}{\hookrightarrow} B \xrightarrow{C} 2^B$

$\sup(c \circ i) = \bigcup_{a \in A} \{a\}.$

$$\begin{tikzcd} A \arrow[r, "F"] \arrow[rr, bend right] & \mathcal{C} \arrow[r, "\text{よ}"] & \mathrm{Set}^{\mathcal{C}^{\circ}} \end{tikzcd}$$

$$F \longmapsto \mathrm{colim}(\text{よ} \circ F)$$

<u>thm</u> $\mathrm{colim}(\text{よ} \circ \pi_X) \cong X_-$

Back to the plan

- $P(A)$ ✓
- $\Pi^{p^{o}} \simeq Low(P)$ ✓ (Groth construction).
- g^{-1} ✓
- $P_!(f)$.
- sup.

② g^{-1}

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, "f"'] & B \arrow[dl, dashed, bend left, "g^{-1}"] \\ 2^{A^\circ} & \end{tikzcd}$$

Def Given a functor $A \xrightarrow{g} B$

its "nerve" $B(g-,-) : B \longrightarrow Set^{A^\circ}$.

$b \longmapsto B(g-, b) : A^\circ \longrightarrow Set$

$a \longmapsto B(ga, b)$.

Rem Sanity check : Posets

$$\begin{tikzcd} P \arrow[r, "g"] \arrow[d, "c"'] & Q \arrow[dl, bend left] \\ \Pi^{P^\circ} & \end{tikzcd} \qquad Q(g-,-) = \text{"} g^{-1} \text{"}.$$

$$Q(g-,q) : P^\circ \longrightarrow \Pi$$

$$p \longmapsto \begin{cases} 1 & gp \leq q \\ 0 & \text{otherwise} \end{cases}$$

$$\text{"}g^{-1}(q)\text{"} = \{ p \in P : gp \leq q \}$$

Def A functor is admissible $A \xrightarrow{g} B$ if for all b in B

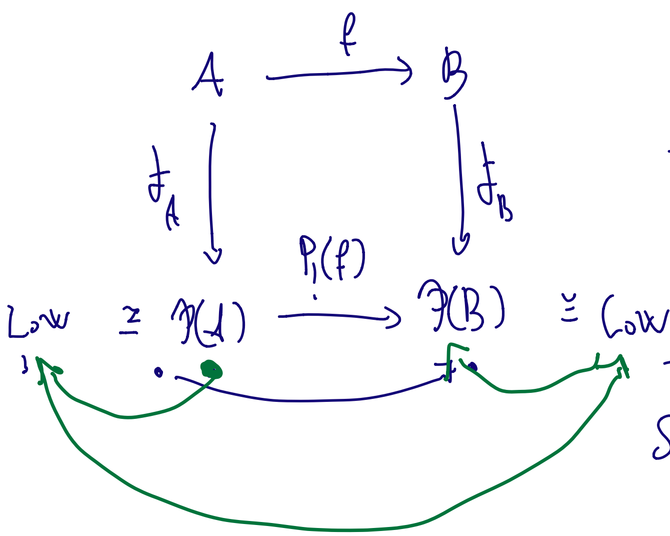
$B(g-,b)$ is a small presheaf.

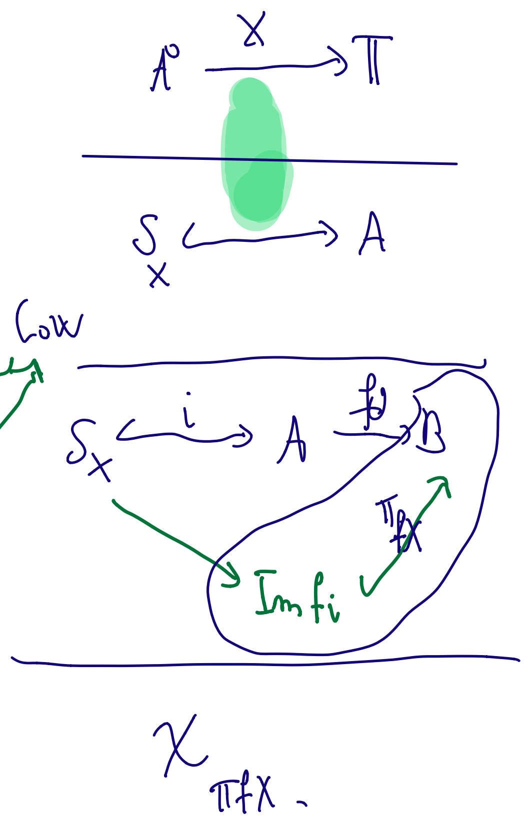
$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d] & B \arrow[dl, dashed] \arrow[ddl, bend left, "{B(g-,-)}"] \\ \mathcal{P}(A) \arrow[d] & \\ \mathrm{Set}^{A^\circ} & \end{tikzcd}$$

Rem functor w/ domain a small category are admissible

Rem functors with arity are admissible.

Cor Accessible functors are admissible.





We do the same!

$$
\begin{tikzcd}
 & A \arrow[r, "f"] \arrow[dl] & B \arrow[dr] & \\
\mathcal{P}(A) \arrow[r, "\mathrm{Elts}"'] \arrow[rrr, bend right, "P_!(f)"'] & \mathrm{Cat}/A \arrow[r, "f\circ -"'] & \mathrm{Cat}/B \arrow[r, "\mathrm{colim}(よ\circ -)"'] & \mathcal{P}(B)
\end{tikzcd}
$$

$$
X \longrightarrow A \xrightarrow{f} B
$$

$P_!(f)$.

Rem

$$
P_!(f)(X) = P_!(f)\Big(\bigcup_{x\in X}\{x\}\Big) = \bigcup_{x\in X} f\{x\}.
$$

$$
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "g"'] & B \\
C \arrow[d, "g^{-1}"'] & \\
\mathcal{P}(A) \arrow[r, "P_!(f)"'] & \mathcal{P}(B) \arrow[uu, dashed, "\sup"']
\end{tikzcd}
$$

<u>thm</u> the following are equivalent

1. B is a cocomplete category

2. $B \xrightarrow{よ} \mathcal{P}(B)$ has a left adjoint.

$$\mathrm{Int} : \mathcal{P}(B) \longrightarrow B.$$

$1 \Rightarrow 2$

$$
\begin{tikzcd}
\mathcal{P}(\mathcal{C}) \arrow[rr, "\mathrm{Int}."] \arrow[dr, "\mathrm{Elts}"'] & & \mathcal{C} \\
& \mathrm{Cat}/\mathcal{C} \arrow[ur, "\mathrm{colim}"'] &
\end{tikzcd}
$$

<u>Lemma</u> If $X : \mathcal{C}^{op} \longrightarrow \mathrm{Set}$ is a small presheaf then there exist a small category D and functors

$$
\begin{tikzcd}
& D \arrow[dl, "j"'] \arrow[dr, "i"] & \\
\mathrm{Elts}(X) \arrow[rr, "\pi_X"'] & & \mathcal{C}
\end{tikzcd}
$$

such that colim π_X exist
iff colim (i) exists
and in that case they coincide

Very good!

<u>thus</u> if f is an admissible functor
and B is cocomplete, then
we can Kan-extend!

$$
\begin{tikzcd}
A \arrow[r, "g"] \arrow[d, "f"'] & B \\
C \arrow[ur, "\mathrm{Lan}_f\, g"] \arrow[d, "B(f-{,}-)"'] & \\
\mathcal{P}(A) \arrow[r, "p_!(g)"] & \mathcal{P}(B) \arrow[uu, "\mathrm{Int}"']
\end{tikzcd}
$$

<u>Rem</u> Left Kan extension
Ran Right Kan extensions.

Rm More general presentation of Kan extensions.

Recall $A \xrightarrow{g} C$ $\forall$ category B

$$[A,B] \xleftarrow{g^*} [C,B]$$
$$f \circ g \longleftarrow\!\shortmid f$$

$\boxed{\text{thm } \text{"}\mathrm{lan}_g\text{"} : [A,B] \longrightarrow [C,B] \quad f \longmapsto \mathrm{lan}_g f \quad \text{is left adjoint to } g^*.}$

Cor / Ex $\mathrm{lan}_{fg}(h) \simeq \mathrm{lan}_f\left(\mathrm{lan}_g(h)\right)$.

Prop if g is fully faithful

$$\left(\mathrm{lan}_g f\right) \circ g \simeq f.$$

AFT (Adjoint functor theorem).

Lemma Let $f: A \to B$ be a cocontinuous functor between cocomplete categories. Then TFAE.

(1) f has a right adj. g

(2) $\mathrm{lan}_f 1$ exists

And in that case they coincide.

$$\begin{tikzcd}
A \arrow[r, "id"] \arrow[d, "f"'] & A \\
B \arrow[ur, bend left, "\mathrm{lan}_f 1"'] &
\end{tikzcd}$$

(Adjoints are lax inverses.)

Thm (AFT) Let $f: A \to B$ be a functor between cocomplete categories. TFAE.

- f is admissible & cocontinuous.
- f has a right adjoint.

$1 \Rightarrow 2$ Yes because $\mathrm{lan}_f 1$ exists.

$2 \Rightarrow 1$ it is cocont because it is a left adj.

$\forall\, b$ in $\mathcal{B}$ $\quad B(f-, b)$ is small?

$$\underline{B(f-, b)} \overset{\text{adjunction}}{\cong} A(-, gb).$$

$$\|$$

$$\underline{よ(gb).}$$

$B(f-, -)$. $B(f, 1)$. $\square$.

$$B(f-, -) : A^{op} \times B \longrightarrow \mathrm{Set}.$$

$$B(f=, -)$$