Course structure

B
- Categories
- (co) limits I
- Adjunctions

Core
- Toy category theory: posets. Summing up & preparing Yoneda
- Yoneda
- (co) limits II

More adv.
- Toy category theory: posets, Preparing for monads & Kan ext
- Monads
- Kan ext
- Monoidal closed & enriched categories

Last time

Categories
Functors
Nat transformations

truly new ideas

- A monoid is a category
- A poset is a category
- opposite category - opposite poset
- there is a category of small categories
- Functor categories, where maps are nats

$$\boxed{\text{Important ex: } \quad \mathrm{Quiv} \simeq \mathrm{Set}^{\cdot \rightrightarrows \cdot}}$$

Lecture 2: Colimits, Epi's, monos.

this is a lesson about standard constructions in CT.

Limits

We start by 3 families of examples: products, equalizer & pull backs
then we give the general definition

Products

· Def of product.

might not exist

the product is the data of the object and the maps!

Ex 1 product of sets
Ex 2 product of spaces. (smallest topology making projections continuous).

Ex 3 vector spaces.
Ex 4 In a poset.
Def of I-ary product.

Ex 5 $\bigwedge$ in a poset.
Ex 6 empty product.

Equalizers

Def of equalizer

Ex 1 in set, equalizers & equations, products and infinitary equations.

$$X \overset{s_i}{\underset{t_i}{\rightrightarrows}} Y_i .$$

Ex 2 in top

Ex 3 Kernel of a homo

$$\mathrm{Ker} f \to A \overset{f}{\underset{0}{\rightrightarrows}} B$$

Ex 4 $Eq(s,t) \cong Ker(s-t)$ in Vect.

Pullbacks

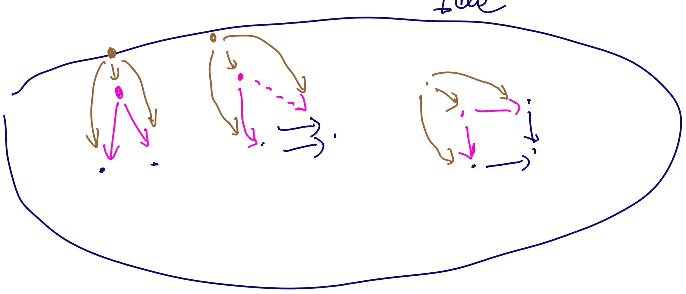
Def of Pullback.

Ex 1 in set, the special example of a fiber.

Ex 2 In a poset we can't distinguish between product and pullback.

Limit

Idea



$\boxed{1}$ Encode the notion of diagram, (small). categories as shapes.

Def (diagram of shape I).

Def (cone of a diagram).

Def (limit cone). Notation: $\lim D$, $\varprojlim D$

One missing example: $(\mathbb{N}, \leq)^{op}$ category & inverse limits.

Example Description of limits in Set.

$$\lim D \cong \mathrm{Set}(1, \lim D)$$

$$\begin{tikzcd} 1 \arrow[r] \arrow[dr, dashed] & \lim D \arrow[d] \arrow[dr] \arrow[drr] & & \\ & D(i) & & \end{tikzcd}$$

$$\cong \{\text{cones on } D \text{ with vertex } 1\}$$

$$\cong \left\{ (x_i)_{i\in I} \quad x_i \in D(i) \;\; \forall i \in I \text{ and } D(u)(x_i) = x_j \;\; \forall u : i \to j. \right\}$$

Rem In a sense this also show how to build limits.
(in Grp, Top, Set, Vect ...).
Not in a poset!

Def Category w/ limits (complete).

Thm Complete $\iff$ products & equalizers

Proof $\Rightarrow$ ✓

$\Leftarrow$ idea: the best in set
the formula we gave is
clearly

$$\lim D \subset \prod_{i \in I} D(i)$$

and going back to our
observation about equalizers:
it is clearly an equalizer
for a family of equations.

$$\lim D = eq\left(\prod_{i \in I} D(i) \underset{t}{\overset{s}{\rightrightarrows}} \prod_{u \in D(i,j)} D(j) \right)$$

where
$$\frac{s : \prod D(i) \longrightarrow \prod D(j)}{s_u : D(i) \xrightarrow[D(u)]{} D(j) \quad \forall\, u : i \to j}$$

and
$$\frac{t : \prod D(i) \longrightarrow \prod D(j)}{t_u : \prod D(i) \xrightarrow{\pi_{D(j)}} D(j) \quad \forall\, u : i \to j}$$

Monos

<u>Def</u> Mono. $\rightrightarrows a \xrightarrow{f} b$

<u>Ex</u> Mono in Set & injective functions.

<u>Ex</u> this can easily be generalized to "points".

<u>Prop</u> $f: a \to b$ is a mono iff

$$
\begin{tikzcd}
a \arrow[r, "1"] \arrow[d, "1"'] \arrow[dr, phantom, "\lrcorner", very near start] & a \arrow[d, "f"] \\
a \arrow[r, "f"'] & b
\end{tikzcd}
$$

Colimits

<u>Def</u> Colimit. Opposite category.

<u>Rem</u> A notational disaster

lim	colim
$\underleftarrow{\lim}$	$\underrightarrow{\lim}$
limit along inverse system	$\underleftarrow{\lim}$
colim inv. sys	$\underrightarrow{\lim}$
inverse limit = lim	direct limit = colim

<u>Ex</u>: Coproducts
Coequalizers
Pushout

<u>Epis</u> <u>Def</u>

Example: $\mathbb{Z} \hookrightarrow \mathbb{Q}$ is an epi in Ring.

Preservation of limits

A functor can preserve / create / reflect limits

Example Forgetful functors preserve limits.

CATEGORY THEORY

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EXERCISES

Leinster. 5.1.39

Exercise 1. The identity functor of a category $\mathbb{1}_C : C \to C$ is a diagram. If it exists, can you describe its limits? And what about its colimit?

Leinster. 5.1.33

Leinster. 5.1.34

Leinster. 5.1.35

Leinster. 5.1.42

Exercise 2. Show that a conservative functor preserving equalizers is also faithful.

Leinster. 5.2.21

Leinster. 5.2.23

Leinster. 5.2.24

Leinster. 5.2.25

Leinster. 5.2.26

Leinster. 5.3.8

Leinster. 5.3.9

Leinster. 5.3.13

Riehl. 3.1.i

Riehl. 3.1.ii

Riehl. 3.1.iii

Riehl. 3.5.i

Date: October 12, 2020.

- the exercises in the red group are mandatory.
- pick at least one exercise from each of the yellow groups.
- pick at least two exercises from each of the blue groups.
- nothing is mandatory in the brown box.
- The riddle of the week. It's just there to let you think about it. It is not a mandatory exercise, nor it counts for your evaluation. Yet, it has a lot to teach.
- useful to deepen your understanding. Take your time to solve it. (May not be challenging at all.)
- measures the difficulty of the exercise. Note that a technically easy exercise is still very important for the foundations of your knowledge.
- It's just too hard.

The label **Leinster** refers to the book **Basic Category Theory**, by *Leinster*.
The label **Riehl** refers to the book **Category Theory in context**, by *Riehl*.