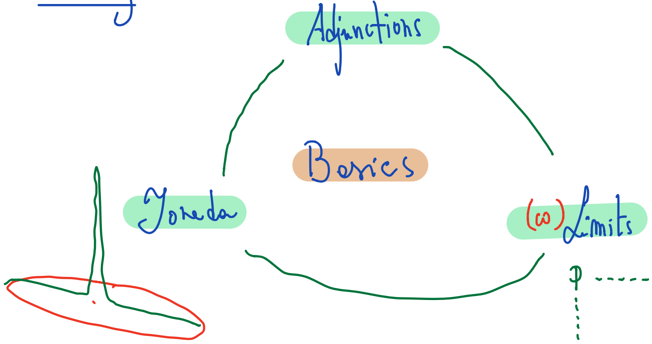
Previously

$A(Lx,y) \cong B(x,Ry)$



$$\begin{tikzcd}
P \arrow[r, dashed] \arrow[d, dashed] & A \arrow[d] \\
C \arrow[r] & B
\end{tikzcd}$$

posetal downgrade

Future topics ← today

Future topics

- Monads
- Kan extensions
- Enriched categories.

today

- Closure operators
- posetal Kan extensions
- Interaction! the downgraded syntax-semantics adj.

① Monads and closure operators.

Def A monad T on a poset is an endofunctor $T : P \longrightarrow P$ s.t.

1) $p \leqslant Tp$

2) $T^2 p \leqslant Tp$

Rmk. T comes from the original name "triple".

Rmk. $T^2 = T$, in fact

$1 \Rightarrow Tp \leqslant T^2 p$

$2 \Rightarrow T^2 p \leqslant Tp$.

Rmk. this happens because a poset does not have many arrows. Monads on a category are much more complex.

$$(T, \eta, \mu).$$

$$1 \xRightarrow{\eta} T \qquad \mu : T^2 \Rightarrow T$$

We specify natural transformations, they obey to axioms.

So today we discuss the simple case.

Q: Have you ever met a monad?

Yes!

Recall that in any vector space V we have

$$\begin{aligned} \mathrm{Span} : \mathcal{P}(V) &\longrightarrow \mathcal{P}(V). \\ A &\longmapsto \mathrm{span}(A). \end{aligned}$$

Similarly, for every group, or algebraic structure

$$\begin{aligned} cl : \mathcal{P}(G) &\longrightarrow \mathcal{P}(G) \\ X &\longmapsto \langle X \rangle. \end{aligned}$$

In model theory

$$X \longrightarrow dcl(X)$$

Recall that a closure operator in topology is.

functoriality

$$c : \mathcal{P}(X) \longrightarrow \mathcal{P}(C)$$

- $X \subset Y \Rightarrow cl(X) \subset cl(Y)$
- $X \subset cl(X).$
- $cl^2(X) = cl(X).$

monad axioms.

additional, preservation of coproducts

$$(K) \quad cl(X \cup Y) = cl(X) \cup cl(Y)$$

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these families are "different", but somehow similar.

Can you see a property that distinguishes the left group from the right one?

Preservation of directed suprema... (algebraic closure operators).

Def An algebra for a monad on a poset is an element p s.t.

$$Tp \leq p.$$

Rmk, again, this is trivialized by the structure of poset.

p is an algebra $\iff$ p is a fixed point.

Def Given a monad T on a poset P we define its poset of algebras $Alg(T)$.
Elements are algebras and morphisms are morphisms of the poset P.

Rem

$$\begin{tikzcd} Alg(T) \arrow[r, hook, "i"'] & P \end{tikzcd}$$

Prop The inclusion has a left adjoint.

$$\begin{tikzcd} cl : P \arrow[r, bend left] & Alg(T) : i \arrow[l, bend left] \end{tikzcd}$$

Proof $Alg(T)(cl\, x, y) \cong P(x, iy)$

$$cl\, x \leq y \iff x \leq y$$

$(\Rightarrow)$ obv

$(\Leftarrow)$ $x \leq y \Rightarrow cl(x) \leq \underset{y}{\underset{\parallel}{cl(y)}}$

y is an algebra

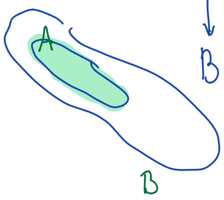
Examples

Monad	Algebras
$cl : X \to X$	closed subspaces
$Span : V \to V$	subspaces
$\langle - \rangle : G \to G$	subgroups

② Kan extensions.

Say that we want to solve an extension problem.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, hook, "i"'] & C \\ B \arrow[ur, dashed, "\bar{f}"'] & \end{tikzcd}$$



$$\begin{tikzcd} i^{-1}(y) \arrow[r] \arrow[d] & A \arrow[r, "f"] \arrow[d, "i"] & C \\ \bullet \arrow[r, "y"] & B & \end{tikzcd}$$

Naive idea

$\underline{\bar{f} = f \circ i^{-1}}$

Problems: • $i^{-1}(x)$ empty

• $i^{-1}(x)$ has many elements, how do I choose?

What if C is a poset?

In that case I could define

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"'] & C \\ B \arrow[ur, "\hat{f}"'] & \end{tikzcd}$$

$$\hat{f}(y) = \begin{cases} \sup_{t \in i^{-1}(y)} f(t). \\ \\ \inf_{t \in i^{-1}(y)} f(t) \end{cases}$$

Let's conceptualize. Assume $\mathcal{C}$ has supreme.

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "i"] \arrow[dd, bend right, "\text{よ}_A"'] & \mathcal{C} \\
B \arrow[d, "i^{-1}"] & \\
\mathcal{P}(A) \arrow[r, "P_!(f)"'] \arrow[uur, dashed, "\mathrm{Lan}_i f"] & \mathcal{P}(\mathcal{C}) \arrow[uu, "\sup"']
\end{tikzcd}$$

left Kan extension

$$\boxed{\mathrm{Lan}_i f = \sup \circ P_!(f) \circ i^{-1}}$$

the "inf" one is the Right Kan extension. It is possible (but not trivial) to write an analogous formula.

this technique solves the problem of "extending functors defined on subcategories".

Rem Size matter !!

Rem Let's study a concrete example. Consider the inclusion

$$\begin{tikzcd} Alg(T) \arrow[r, hook, "i"'] & P \arrow[l, dashed, bend right, "T"'] \end{tikzcd}$$

$$\begin{tikzcd} Alg(T) \arrow[r, equal] \arrow[d, hook, "i"'] & Alg(T) \\ P \arrow[ur, "\mathrm{ran}_i(1)"'] & \end{tikzcd}$$

$$\mathrm{ran}_i(1)(p) = \inf_{p \leq i(x)} x$$

$$= \inf_{Tp \leq x} x$$

$$= T(p).$$

$$\boxed{\mathrm{ran}_i(1) = T.}$$

Rem $\mathrm{ran}_i\, i$ is always a closure operator

So if P is a complete poset, we obtain a construction

$$\begin{tikzcd} \mathrm{Pos}/P \arrow[r, bend left, "\mathrm{ran}_{(-)}(-)"'] & \text{Monads over } P \end{tikzcd}$$

And in the opposite direction

$$\begin{tikzcd} \mathrm{Pos}/P & \text{Monads over } P \arrow[l, bend left, "Alg(-)"'] \end{tikzcd}$$

This is called the "structure-semantics" adjunction and is useful to recognize those

$$\mathcal{C} \xrightarrow{i} P$$

such that there is a monad $T : P \longrightarrow P$, such that

$$\begin{tikzcd} \mathcal{C} \arrow[rr, dotted] \arrow[dr, "i"'] & & Alg(T) \arrow[dl, "j"] \\ & P & \end{tikzcd}$$

$\mathcal{C}$ is isomorphic to $Alg(T)$.

If it happens i is "monadic".

Exemple : $\mathcal{C}\ell(X) \xhookrightarrow{i} X$

In fact, if it exist

$$\begin{tikzcd} \mathcal{C} \arrow[d, "i"'] \\ P \end{tikzcd} = \mathrm{Alg}(\mathrm{ran}_i\, i).$$

<u>thm</u> (Beck monadicity silly version)

$\mathcal{C} \xrightarrow{i} P$, P complete is monadic

iff

1. $\mathcal{C}$ is complete
2. i preserves inf.
3. i is conservative (injective)

<u>Proof</u> $(\Leftarrow)$

③ obv.

② i is a right adj.

① $\mathcal{C}$ is closed under inf in P.

$(\Rightarrow)$ Define $\mathrm{ran}_i(1) = cl$

Define $T = i \circ cl$.